Regularization estimates and hydrodynamical limit for the Landau equation

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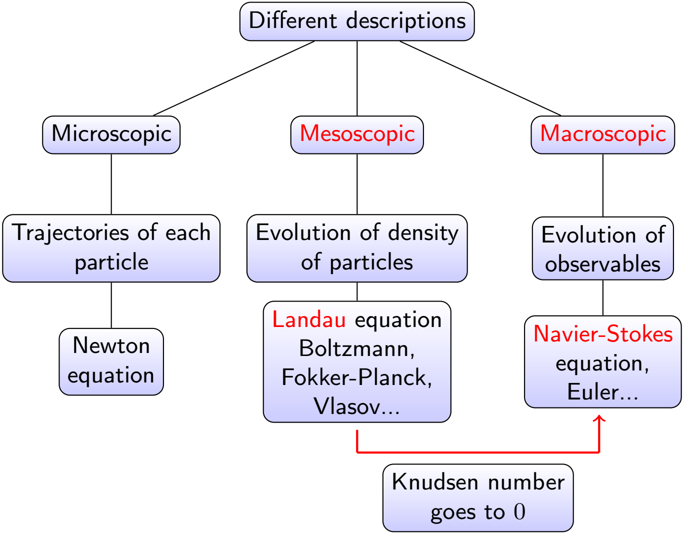
In a particle system (gas)



The Landau equation

- Landau (1936) : Kinetic model in plasma physics that describes the evolution of the density function $f(t,x,v)$, $t \in \mathbb{R}^+$ the time, $x \in \mathbb{T}^3$ the position and $v \in \mathbb{R}^3$ the velocity.

Landau equation

$$\partial_t f + v \cdot \nabla_x f = Q(f,f)$$

- Q is the Landau collision operator (bilinear operator and acts only on variable v) :

$$Q(f,g)(v) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a(v - v_\star)[g(v_\star)\nabla_v f(v) - f(v)\nabla_v g(v_\star)] \, dv_\star \right\},$$

$$a_{i,j}(z) = |z|^{\gamma+2} \underbrace{\left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right)}_{\text{projection onto } z^\perp} .$$

- $\gamma \in [-2,1] \leftrightarrow$ hard potentials, Maxwellian molecules and moderately soft potentials.
- $\gamma \in (-3,-2) \leftrightarrow$ very soft potentials.
- $\gamma = -3 \leftrightarrow$ Coulombian potential.

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Remarque

For $g(v) = \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$, *we have*

- $Q(f,\mu)(v) = \nabla_v \cdot \{(a *_v \mu)\nabla_v f - (b *_v \mu) f\}, \quad b_i(v) = \sum_j \partial_j a_{ij}(v)$
- *It looks like the Fokker-Planck operator :*

$$L_{FP}(f) = \nabla_v \cdot \{\nabla_v f + v f\}$$

Basic properties

- Conservation of mass, momentum and energy :

$$\int_{\mathbb{R}^3} Q(f,f) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} \mathrm{d}v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Entropy of the system :

$$H(f) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f \,\log f \; \mathrm{d}x\mathrm{d}v.$$

- Entropy dissipation :

$$D(f) := -\int_{\mathbb{R}^3} Q(f,f) \,\log f \; \mathrm{d}v \geqslant 0.$$

- H-Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = -\int_{\mathbb{T}^3} D(f) \; \mathrm{d}x \leqslant 0.$$

- $D(f) = 0 \Longleftrightarrow f$ (local Maxwellian).
- $Q(\mu,\mu) = 0.$

The normalized Maxwellian

$$\mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$$

The rescaled Landau equation

- The model :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f,f), & (t,x,v) \in \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3 \\ f_{|t=0} = f_0, \end{cases}$$

- The Knudsen number : $1/\varepsilon$ is the average number of collisions for each particle per unit time
- Rescaling and perturbation :
  - $f_\varepsilon(t,x,v) = f\big(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}, v\big)$
  - $f_\varepsilon(t,x,v) = \mu + \varepsilon \mu^{1/2} g_\varepsilon(t,x,v)$
- The perturbative system :

$$\begin{cases} \partial_t g_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x g_\varepsilon - \frac{1}{\varepsilon^2} \mathcal{L} g_\varepsilon = \frac{1}{\varepsilon} \Gamma(g_\varepsilon, g_\varepsilon) \\ g_{\varepsilon\,|t=0} = g_{\varepsilon,0} \end{cases} \tag{1}$$

- $\Gamma(g_1, g_2) = \mu^{-1/2} Q(\mu^{1/2} g_1, \mu^{1/2} g_2)$
- $\mathcal{L}$ is the homogeneous linearized Landau operator :
  - $\mathcal{L}$ acts only in variable v
  - $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^3_v)$
  - $\mathcal{L}$ is a negative operator $L^2(\mathbb{R}^3_v)$

- $\mathcal{L} = \underbrace{\mathcal{L}_1}_{\text{diffusion part}} + \underbrace{\mathcal{L}_2}_{\text{compact part}}$
- $\mathcal{L}_1 g = \Gamma(\sqrt{\mu}, g), \quad \mathcal{L}_2 g = \Gamma(g, \sqrt{\mu})$
- The diffusion part $\mathcal{L}_1$:

$$\mathcal{L}_1 g = \nabla_v \cdot [\mathbf{A}(v)\nabla_v g] - \Big(\mathbf{A}(v)\frac{v}{2}\cdot\frac{v}{2}\Big)g + \nabla_v \cdot \Big[\mathbf{A}(v)\frac{v}{2}\Big]g.$$

  - $\mathbf{A}(v) = (\overline{a}_{ij}(v))_{1\leqslant i,j\leqslant 3}$ is a symmetric matrix with

$$\overline{a}_{ij} = a_{ij} *_v \mu.$$

  - $\mathbf{A}(v)$ is written as follows : $\mathbf{A}(v) = B^{\mathsf{T}}(v)B(v)$
- The compact part $\mathcal{L}_2$:

$$\mathcal{L}_2 g = -\mu^{-1/2}\partial_i \left\{\mu\left[a_{ij} *_v \left\{\mu^{1/2}\left[\partial_j g + \frac{v_j}{2}g\right]\right\}\right]\right\}.$$

- $\mathcal{N}(\mathcal{L}) = \text{Span}\left\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2\mu\right\}.$

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} g_{\varepsilon,0}(x,v)\mu^{1/2}(v)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}\mathrm{d}x\mathrm{d}v = \begin{pmatrix}0\\0\\0\end{pmatrix} \quad (IC)$$

Notations

- $H^1_{v,\star}$-norm defined by

$$\|g\|^2_{H^1_{v,\star}} := \|\langle v\rangle^{\frac{\gamma}{2}+1} g\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}} P_v \nabla_v g\|^2_{L^2_v} + \|\langle v\rangle^{\frac{\gamma}{2}+1}(I - P_v)\nabla_v g\|^2_{L^2_v},$$

where $\langle v\rangle = (1+|v|^2)^{1/2}$ and P_v is the projection on v, i.e. $P_v\eta = \left(\eta\cdot\frac{v}{|v|}\right)\frac{v}{|v|}$

- Sobolev-type spaces $\mathscr{X} = \mathscr{H}^3_x(L^2_v)$, $\mathscr{Y} = \mathscr{H}^3_x(H^1_{v,\star})$

$$\|f\|^2_{\mathscr{X}} := \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|^2_{L^2_{x,v}} + \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla^2_x f\|^2_{L^2_{x,v}} + \|\nabla^3_x f\|^2_{L^2_{x,v}}$$

$$\|f\|^2_{\mathscr{Y}} := \|\langle v\rangle^{3(\frac{\gamma}{2}+1)} f\|^2_{L^2_x(H^1_{v,\star})} + \|\langle v\rangle^{2(\frac{\gamma}{2}+1)} \nabla_x f\|^2_{L^2_x(H^1_{v,\star})} + \|\langle v\rangle^{\frac{\gamma}{2}+1} \nabla^2_x f\|^2_{L^2_x(H^1_{v,\star})} + \|\nabla^3_x f\|^2_{L^2_x(H^1_{v,\star})}$$

- $\mathbf{A}(v)$ is written as follows : $\mathbf{A}(v) = B^{\mathsf{T}}(v)B(v)$
  - $\tilde{\nabla}_v := B(v)\nabla_v$
  - $\tilde{\nabla}_x := B(v)\nabla_x$.
- $g = \Pi_0 g + (I - \Pi_0)g$; Π_0 is the orthogonal projection to $\mathcal{N}$
- $\Pi_0 g$: The fluid/macroscopic part
- $g^\perp := (I - \Pi_0)g$: the kinetic or microscopic part

Existence and regularization of the solution

Théorème (Carrapatoso-R.-Tristani)

There is $\eta_0 > 0$ *small enough such that for any* $\varepsilon \in (0,1)$*, if* $g_{\varepsilon,0} \in \mathscr{X}$ *satisfies* (IC) *and* $\|g_{\varepsilon,0}\|_{\mathscr{X}} \leqslant \eta_0$ *then the following holds :*

(i) There is a unique global solution $g_\varepsilon \in L^\infty(\mathbb{R}^+;\mathscr{X}) \cap L^2(\mathbb{R}^+;\mathscr{Y})$ *à (1) associated to the initial data* $g_{\varepsilon,0}$*, which verifies moreover*

$$\sup_{t\geqslant 0} e^{2\sigma t}\|g_\varepsilon(t)\|^2_{\mathscr{X}} + \frac{1}{\varepsilon^2}\int_0^\infty e^{2\sigma t}\|(g_\varepsilon(t))^\perp\|^2_{\mathscr{Y}}\,\mathrm{d}t + \int_0^\infty e^{2\sigma t}\|g_\varepsilon(t)\|^2_{\mathscr{Y}}\,\mathrm{d}t$$
$$\leqslant C\|g_{\varepsilon,0}\|^2_{\mathscr{X}},$$

where $\sigma, C > 0$ *are independent of* ε*.*

(ii) In addition, the solution satisfies the following regularization estimates

$$\|g_\varepsilon(t)\|_{\mathscr{Y}} \leqslant C\frac{e^{-\sigma t}}{\min(1,\sqrt{t})}\|g_{\varepsilon,0}\|_{\mathscr{X}}, \quad \forall\, t > 0,$$

where $C > 0$ *is independent of* ε*.*

Strategy of the proof :

For (i) :

1. We introduce the norm $|||\cdot|||_{L^2_{x,v}}$ as follows :

$$|||f|||^2_{L^2_{x,v}} = \|f\|^2_{L^2_{x,v}} + \varepsilon \sum_{i=1}^{3} \eta_i \left\langle \partial_{x_i} \Delta_x^{-1} \Pi_0 f, \widetilde{\Pi}_i f \right\rangle_{L^2_x}$$

- The constants η_i are chosen to be small enough
- The inverse laplacian Δ_x^{-1}
- $\widetilde{\Pi}_i : L^2_{x,v} \to L^2_x$ is some suitable moment operator
- We have $|||\cdot|||_{L^2_{x,v}} \sim \|\cdot\|_{L^2_{x,v}}$ (uniformly in ε).

2. We introduce the norm $|||\cdot|||_{\mathscr{X}}$ as follows : for $\delta \in (0,1)$,

$$|||f|||^2_{\mathscr{X}} := \sum_{i=0}^{2} \left(\delta \|\langle v\rangle^{(3-i)(\frac{\gamma}{2}+1)} \nabla_x^i f^\perp\|^2_{L^2_{x,v}} + |||\nabla_x^i f|||^2_{L^2_{x,v}} \right) + |||\nabla_x^3 f|||^2_{L^2_{x,v}}.$$

3. The solution g_ε verifies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |||g_\varepsilon|||^2_{\mathscr{X}} = \langle\langle \Lambda_\varepsilon g_\varepsilon, g_\varepsilon \rangle\rangle_{\mathscr{X}} + \frac{1}{\varepsilon} \langle\langle \Gamma(g_\varepsilon, g_\varepsilon), g_\varepsilon \rangle\rangle_{\mathscr{X}}$$

- $\Lambda_\varepsilon := \frac{1}{\varepsilon^2}\mathcal{L} - \frac{1}{\varepsilon} v \cdot \nabla_x$.

4. Hypocoercivity estimate : We choose $\delta > 0$ small enough to show that

$$\langle\!\langle \Lambda_\varepsilon g_\varepsilon, g_\varepsilon \rangle\!\rangle_{\mathscr{X}} \leqslant -\sigma |\!|\!| g_\varepsilon |\!|\!|^2_{\mathscr{X}} - \kappa \| g_\varepsilon \|^2_{\mathscr{Y}} - \frac{\kappa}{\varepsilon^2} \| g_\varepsilon^\perp \|^2_{\mathscr{Y}} \quad (\sigma, \kappa > 0).$$

5. We show

$$\frac{1}{\varepsilon} \langle\!\langle \Gamma(g_\varepsilon, g_\varepsilon), g_\varepsilon \rangle\!\rangle_{\mathscr{X}} \leqslant \frac{C}{\varepsilon} |\!|\!| g_\varepsilon |\!|\!|_{\mathscr{X}} \| g_\varepsilon \|_{\mathscr{Y}} \| (g_\varepsilon)^\perp \|_{\mathscr{Y}} \quad (C > 0).$$

6. We introduce $g_\varepsilon^\sigma = e^{\sigma t} g_\varepsilon$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\!|\!| g_\varepsilon^\sigma |\!|\!|^2_{\mathscr{X}} \leqslant -\left(\kappa - C |\!|\!| g_\varepsilon^\sigma |\!|\!|^2_{\mathscr{X}} \right) \| g_\varepsilon^\sigma \|^2_{\mathscr{Y}} - \frac{\kappa}{2\varepsilon^2} \| (g_\varepsilon^\sigma)^\perp \|^2_{\mathscr{Y}}.$$

4. Hypocoercivity estimate : We choose $\delta > 0$ small enough to show that

$$\langle\langle \Lambda_\varepsilon g_\varepsilon, g_\varepsilon \rangle\rangle_{\mathscr{X}} \leqslant -\sigma |||g_\varepsilon|||_{\mathscr{X}}^2 - \kappa \|g_\varepsilon\|_{\mathscr{Y}}^2 - \frac{\kappa}{\varepsilon^2} \|g_\varepsilon^\perp\|_{\mathscr{Y}}^2 \quad (\sigma, \kappa > 0).$$

5. We show

$$\frac{1}{\varepsilon} \langle\langle \Gamma(g_\varepsilon, g_\varepsilon), g_\varepsilon \rangle\rangle_{\mathscr{X}} \leqslant \frac{C}{\varepsilon} |||g_\varepsilon|||_{\mathscr{X}} \|g_\varepsilon\|_{\mathscr{Y}} \|(g_\varepsilon)^\perp\|_{\mathscr{Y}} \quad (C > 0).$$

6. We introduce $g_\varepsilon^\sigma = e^{\sigma t} g_\varepsilon$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |||g_\varepsilon^\sigma|||_{\mathscr{X}}^2 \leqslant -\left(\kappa - C |||g_\varepsilon^\sigma|||_{\mathscr{X}}^2\right) \|g_\varepsilon^\sigma\|_{\mathscr{Y}}^2 - \frac{\kappa}{2\varepsilon^2} \|(g_\varepsilon^\sigma)^\perp\|_{\mathscr{Y}}^2 .$$

Pour (ii) :

1. We introduce the functional $\mathscr{U}_\varepsilon$ as follows :

$$\mathscr{U}_\varepsilon(t, g_\varepsilon) = |||g_\varepsilon|||_{\mathscr{X}}^2 + \alpha_1 t \left(\|\tilde{\nabla}_v (g_\varepsilon)^\perp\|_{\mathscr{X}}^2 + K \|\langle v\rangle^{\frac{\gamma}{2}+1} (g_\varepsilon)^\perp\|_{\mathscr{X}}^2 \right)$$
$$+ \varepsilon \alpha_2 t^2 \left\langle \tilde{\nabla}_v g_\varepsilon, \tilde{\nabla}_x g_\varepsilon \right\rangle_{\mathscr{X}} + \varepsilon^2 \alpha_3 t^3 \left(\|\tilde{\nabla}_x g_\varepsilon\|_{\mathscr{X}}^2 + K \|\langle v\rangle^{\frac{\gamma}{2}} \nabla_x g_\varepsilon\|_{\mathscr{X}}^2 \right)$$

- $K > 0$ and $0 < \alpha_3 \ll \alpha_2 \ll \alpha_1 \ll 1$ so that $\alpha_2 \leqslant \sqrt{\alpha_1 \alpha_3}$.

2. We choose $\alpha_i > 0$ for $i = 1 \cdots 3$ small enough and $K > 0$ large enough to show that

$$\mathscr{U}_\varepsilon(t, g_\varepsilon) \leqslant \mathscr{U}_\varepsilon(0).$$

3. We use that $t \|g_\varepsilon\|_{\mathscr{Y}}^2 \lesssim \mathscr{U}_\varepsilon(t, g_\varepsilon)$.

Hydrodynamical limit

- The hydrodynamical limit of (1) as ε goes to zero is the incompressible Navier-Stokes-Fourier system associated with the Boussinesq equation which writes

$$\begin{cases} \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta, \\ \nabla_x \cdot u = 0, \\ \rho + \theta = 0. \end{cases}$$

- θ : temperature
- ρ : density
- p : pressure
- u : velocity vector field
- ν : viscocity
- κ : heat conductivity

- The coefficients ν and κ are determined by $\mathcal{L}$.

- We introduce g_0 as follows :

$$g_0(x,v) := \sqrt{\mu}(v)\left(\rho_0(x) + u_0(x)\cdot v + \frac{|v|^2-3}{2}\theta_0(x)\right)$$

  - $\nabla_x \cdot u_0 = 0$ and $\rho_0 + \theta_0 = 0$
  - (ρ_0, u_0, θ_0) are the associated macroscopic quantities
  - g_0 is called well-prepared data.

Remarque

There exists $\eta_1 > 0$ such that if g_0 satisfies $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$, then there exists $(\rho, u, \theta) \in H^3_x$ a solution of the incompressible Navier-Stokes-Fourier system defined on $\mathbb{R}^+$.

- We define the kinetic distribution lying in $\mathcal{N}(\mathcal{L})$:

$$g(t,x,v) := \sqrt{\mu}(v)\left(\rho(t,x) + u(t,x)\cdot v + \frac{|v|^2-3}{2}\theta(t,x)\right)$$

  - (ρ, u, θ) are the associated macroscopic quantities and a solution of the incompressible Navier-Stokes-Fourier system.

Théorème (Carrapatoso-R.-Tristani)

Let $g_{\varepsilon,0} \in \mathscr{X}$ *for* $\varepsilon \in (0,1)$ *such that* $\|g_{\varepsilon,0}\|_{\mathscr{X}} \leqslant \eta_0$ *and verifies* (IC)*. Consider also* $g_0 \in \mathscr{X}$ *such that* $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$ *and verifies* (IC)*.*
There exists $\eta_2 \in (0, \min(\eta_0, \eta_1))$ *such that if* $\max\left(\|g_{\varepsilon,0}\|_{\mathscr{X}}, \|g_0\|_{\mathscr{X}}\right) \leqslant \eta_2$ *and*

$$\|g_{\varepsilon,0} - g_0\|_{\mathscr{X}} \xrightarrow[\varepsilon\to 0]{} 0,$$

then we have

$$\|g_\varepsilon - g\|_{L^\infty_t(\mathscr{X})} \xrightarrow[\varepsilon\to 0]{} 0.$$

If

$$\|\Pi_0 g_{\varepsilon,0} - g_0\|_{\mathscr{X}} \xrightarrow[\varepsilon\to 0]{} 0,$$

then we have

$$\|g_\varepsilon - g\|_{L^1_t(\mathscr{Y}) + L^\infty_t(\mathscr{X})} \xrightarrow[\varepsilon\to 0]{} 0.$$

Théorème (Quantitative estimates)

Let $g_{\varepsilon,0} \in \mathscr{X}$ *for* $\varepsilon \in (0,1)$ *such that* $\|g_{\varepsilon,0}\|_{\mathscr{X}} \leqslant \eta_0$ *and verifies* (IC)*. Consider also* $g_0 \in H_x^{3+\delta}L_v^2$ *for some* $\delta \in [0,1/2]$ *such that* $\|g_0\|_{\mathscr{X}} \leqslant \eta_1$ *and verifies* (IC)*.*

There exists $\eta_2 \in (0, \min(\eta_0,\eta_1))$ *such that if* $\max\left(\|g_{\varepsilon,0}\|_{\mathscr{X}}, \|g_0\|_{\mathscr{X}}\right) \leqslant \eta_2$*, then we have*

$$\|g_\varepsilon - g\|_{L_t^\infty(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H_x^{3+\delta}L_v^2}, \|g_{\varepsilon,0}\|_{\mathscr{X}}\right) + \|g_{\varepsilon,0} - g_0\|_{\mathscr{X}}$$

and

$$\|g_\varepsilon - g\|_{L_t^1(\mathscr{Y})+L_t^\infty(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H_x^{3+\delta}L_v^2}, \|g_{\varepsilon,0}\|_{\mathscr{X}}\right) + \|\Pi_0 g_{\varepsilon,0} - g_0\|_{\mathscr{X}}$$

where $C\left(\|g_0\|_{H_x^{3+\delta}L_v^2}, \|g_{\varepsilon,0}\|_{\mathscr{X}}\right)$ *is a contant only depending on* $\|g_0\|_{H_x^{3+\delta}L_v^2}$ *and* $\|g_{\varepsilon,0}\|_{\mathscr{X}}$ *.*

Strategy of the proof :

Step 1 :

- $U^\varepsilon(t)$ is the semigroup associated to the operator Λ_ε.
- From Duhamel formula, we have that

$$g_\varepsilon(t) = U^\varepsilon(t) g_{\varepsilon,0} + \underbrace{\frac{1}{\varepsilon}\int_0^t U^\varepsilon(t-s)\Gamma\big(g_\varepsilon, g_\varepsilon\big)(s)\,\mathrm{d}s}_{\Psi^\varepsilon(t)(g_\varepsilon, g_\varepsilon)}.$$

- In some sense, we have $U^\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} U(t)$ et $\Psi^\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} \Psi(t)$ and the limit g is written as follows [Bardos-Ukai, 1991] :

$$g(t) = U(t)g_0 + \Psi(t)(g, g).$$

- We introduce [Gallagher-Tristani, 2020]

$$h_\varepsilon := g_\varepsilon - g.$$

- L^∞_t-estimate : h_ε verifies

$$\begin{aligned} h_\varepsilon &= (U^\varepsilon(t) - U(t))g_0 + U^\varepsilon(t)(g_{\varepsilon,0} - g_0) + (\Psi^\varepsilon(t) - \Psi(t))(g, g) \\ &\quad + \Psi^\varepsilon(t)((g_\varepsilon)^\perp, (g_\varepsilon)^\perp) + \Psi^\varepsilon(t)((g_\varepsilon)^\perp, \Pi_0 g_\varepsilon) + \Psi^\varepsilon(t)(\Pi_0 g_\varepsilon, (g_\varepsilon)^\perp) \\ &\quad + \Psi^\varepsilon(t)(\Pi_0 h_\varepsilon, \Pi_0 g_\varepsilon) + \Psi^\varepsilon(t)(g, \Pi_0 h_\varepsilon). \end{aligned}$$

Étape 2 :

- On montre pour tout $t \geqslant 0$ et $\delta \in [0, 1/2]$,

$$\|(U^\varepsilon(t) - U(t))g_0\|_{\mathscr{X}} + \|(\Psi^\varepsilon(t) - \Psi(t))(g, g)\|_{\mathscr{X}} \lesssim \varepsilon^\delta C\Big(\|g_0\|_{H^{3+\delta}_x L^2_v}\Big).$$

- On montre pour tout $t \geqslant 0$

$$\|U^\varepsilon(t)(g_{\varepsilon,0} - g_0)\|_{L^\infty_t(\mathscr{X})} \lesssim \|g_{\varepsilon,0} - g_0\|_{\mathscr{X}}.$$

- We show for all $t \geqslant 0$

$$\|\Psi^\varepsilon(t)((g_\varepsilon)^\perp, (g_\varepsilon)^\perp)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)((g_\varepsilon)^\perp, \Pi_0 g_\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(\Pi_0 g_\varepsilon, (g_\varepsilon)^\perp)\|_{\mathscr{X}} \lesssim \sqrt{\varepsilon}\|g_{\varepsilon,0}\|^2_{\mathscr{X}}.$$

- We show for all $t \geqslant 0$

$$\|\Psi^\varepsilon(t)(\Pi_0 h_\varepsilon, \Pi_0 g_\varepsilon)\|_{\mathscr{X}} + \|\Psi^\varepsilon(t)(g, \Pi_0 h_\varepsilon)\|_{\mathscr{X}} \lesssim \left(\|g_{\varepsilon,0}\|_{\mathscr{X}} + \|g_0\|_{\mathscr{X}}\right)\|h_\varepsilon\|_{L^\infty_t(\mathscr{X})}.$$

- By taking $\|g_0\|_{\mathscr{X}}$ and $\|g_{\varepsilon,0}\|_{\mathscr{X}}$ small enough, we obtain

$$\|h_\varepsilon\|_{L^\infty_t(\mathscr{X})} \lesssim \varepsilon^\delta C\left(\|g_0\|_{H^{3+\delta}_x L^2_v}, \|g_{\varepsilon,0}\|_{\mathscr{X}}\right) + \|g_{\varepsilon,0} - g_0\|_{\mathscr{X}}.$$

1) Some existing results of weak convergence in the framework of strong solutions :

- From Landau equation to incompressible Navier-Stokes : [Rachid, 2021]
- From Boltzmann equation without cutoff to incompressible Navier-Stokes : [Jiang-Xu-Zhao, 2018]
- From Boltzmann equation with cutoff to incompressible Navier-Stokes : [Briant, 2015], [Briant-Merino-Mouhot, 2019], [Guo, 2006]

2) Some existing results of strong convergence in the framework of strong solutions :

- From Boltzmann equation with cutoff to incompressible Navier-Stokes : [Gallagher-Tristani, 2020]

3) Some existing results of weak convergence in the framework of weak solutions :

- The renormalized solutions for the Boltzmann equation (from DiPerna-Lions) and the Leray solutions for the Navier-Stokes equations : Bardos, Golse, Levermore, Lions, Masmoudi, Saint-Raymond...

Conclusion and Perspective

Conclusion :

- We have obtained a strong convergence to the Navier-Stokes solutions from the Landau equation.

Perspectives :

- Is it possible to extend the previous result to the case when $\gamma \in [-3, -2[$?
- What about the limit of Vlasov-Poisson-Landau ?

Thanks for your attention !

K. Carrapatoso, M. Rachid, and I. Tristani. Regularization estimates and hydrodynamical limit for the Landau equation, 2021, arXiv : 2107.12044