A resolvent method for Metastability

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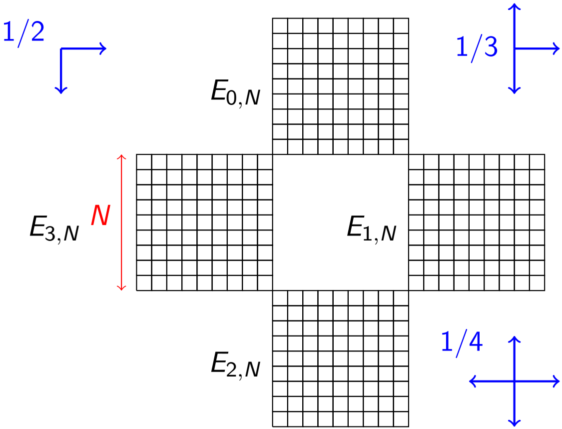
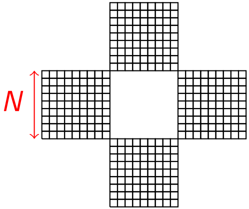
A model: symmetric random walk

$$\eta \in E_N \qquad \eta_N(t) \qquad \pi_N(\eta) = Z_N^{-1} \deg(\eta)$$



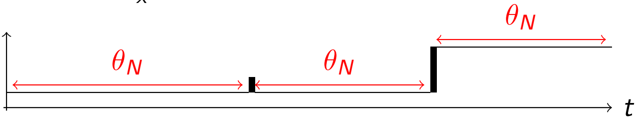
$$\eta_N(t\,\theta_N) \sim X_N(t) \in S = \{0, 1, 2, 3\}$$

Coarse-graining



$$t^N_{\mathrm{mix}} \sim N^2 \qquad t^N_{\mathrm{hit}} \sim N^2 \log N \;\; (d=2) \qquad t^N_{\mathrm{hit}} \sim N^d \;\; (d \geq 3)$$

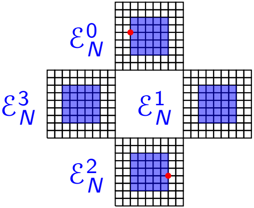
$$\Psi_N(\eta) = \sum_x x \, \chi_{E_{x,N}}(\eta) \qquad Y_N(t) = \Psi_N(\,\eta_N(t\,\theta_N)\,)$$



$$Y_N(t) \longrightarrow Y(t) \quad \mathrm{f.d.d.}$$

path surgery

The trace process



$\eta(t)$ $E-$ valued MC

$$F \subsetneq E \quad T_F(t) = \int_0^t \chi_F(\eta(s))\, ds$$

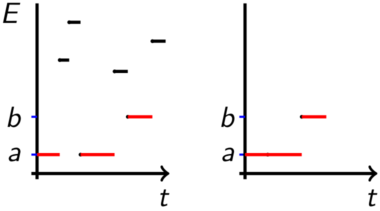
$$S_F(t) = \sup\{s \geq 0 : T_F(s) \leq t\}$$

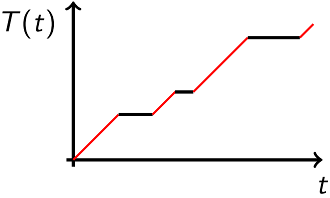
$S_F(t)$ increasing stopping times

$$\eta^F(t) = \eta(S_F(t))$$

$\eta^F(t)$ Markov chain trace process

$$R^F(\eta, \xi) = \lambda(\eta)\, \mathbb{P}_\eta[\tau_F^+ = \tau_\xi]$$





Model reduction

$$\mathcal{E}_n = \bigcup_{x\in S} \mathcal{E}_N^x \quad \Delta_N = E_N \setminus \mathcal{E}_N \quad \Phi_N : \mathcal{E}_N \to S$$

$$\eta^{\mathcal{E}}(t) \ \text{ trace of } \eta(t) \text{ on } \mathcal{E}_N$$

$$X_N(t) = \Phi_N(\eta^{\mathcal{E}}(t)) \quad \text{slow variable} \qquad \text{Hidden Markov}$$

Definition: Beltrán, L. (10)

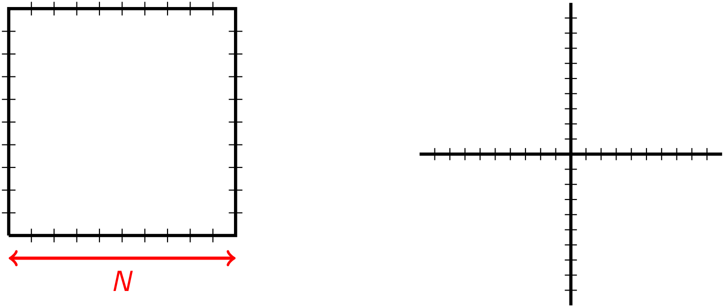
Condition $\mathfrak{C}$: $\boldsymbol{X}_N(t) = X_N(t\theta_N) \longrightarrow \boldsymbol{X}(t)$

Condition $\mathfrak{D}$: $\lim_{N\to\infty} \sup_{\eta\in\mathcal{E}_N} \mathbb{E}_\eta^N\Big[\int_0^t \mathbf{1}\{\eta(s\theta_N) \in \Delta_N\}\, ds\Big] = 0$

L., Loulakis, Mourragui ('16)

- $\Psi_N(\eta(t\theta_N)) \longrightarrow \boldsymbol{X}(t)$ f.d.d
- $\eta_N \in \mathcal{E}_N^j$ $\quad \delta_{\eta_N} P_N(t\,\theta_N) \longrightarrow \sum_k p_t(j,k)\,\pi_k$

Loss of memory: counter-examples



Theorem (D. Marcondes, I. Seo)

$$E_N \qquad \mathcal{E}_n = \bigcup_{x\in S} \mathcal{E}_N^x \qquad \Delta_N = E_N \setminus \mathcal{E}_N$$

Condition $\mathfrak{R}$: $\exists\, S$-valued chain (L): $\forall\, f : S \to \mathbb{R}$, $\lambda > 0$,

$$G_N : E_N \to \mathbb{R} \qquad G_N(\eta) \;=\; \sum_{x\in S} [\,(\,\lambda - L\,)\,f\,](x)\,\chi_{\mathcal{E}_N^x}(\eta)$$

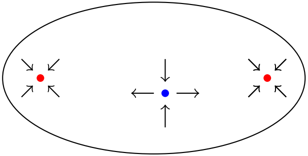
$$(\,\lambda \;-\; \theta_N\,\mathfrak{L}_N\,)\,F_N \;=\; G_N$$

$$\lim_{N\to\infty}\,\max_{x\in S}\,\sup_{\eta\in\mathcal{E}_N^x}\,\big|\,F_N(\eta) \;-\; f(x)\,\big| \;=\; 0$$

Theorem: Assume $\mathfrak{D}$. Then, $\mathfrak{C}$ if and only if $\mathfrak{R}$.

Comments

- Condition $\mathfrak{R}$ does not rely on the stationary state



$$\dot{X}(t) = b(X(t)) \qquad b = \nabla V$$
$$dX_\epsilon(t) = b(X_\epsilon(t)) + \sqrt{2\epsilon}\, dW(t)$$
$$\mathcal{L}_\epsilon f = b \cdot \nabla f + \epsilon \Delta f$$
$$(\lambda - \mathcal{L}_\epsilon) F_\epsilon = G$$

- The set Δ_N

- Condition $\mathfrak{R}$ holds for many dynamics
  - Ising, Potts, Blume-Capel at low temperature, ABC model
  - Random walks and diffusions
  - Condensing zero-range processes
  - Inclusion processes
  - Random polymers

Potential theory interlude

The Dirichlet problem

$$\begin{cases} \mathcal{L} h = 0 & \mathcal{E} \setminus \mathcal{A} \\ h = \varphi & \mathcal{A} \end{cases}$$

$$\mu \qquad \langle f \,,\, \mathcal{L}\, g \rangle \;=\; \langle \mathcal{L}\, f \,,\, g \rangle$$

$$\mathcal{D}(f) \;=\; \langle f \,,\, (-\mathcal{L})\, f \rangle \;\geq\; 0$$

$$\inf_f \mathcal{D}(f) \qquad f\, \chi_{\mathcal{A}} \;=\; \varphi\, \chi_{\mathcal{A}}$$

Capacities

$$\mathcal{A} \cap \mathcal{B} = \varnothing \qquad \begin{cases} \mathfrak{L} h = 0 & (\mathcal{A} \cup \mathcal{B})^c \\ h = \chi_A & \mathcal{A} \cup \mathcal{B} \end{cases}$$

$$h_{\mathcal{A},\mathcal{B}} \qquad h_{\mathcal{A},\mathcal{B}}(\eta) \;=\; \mathbf{P}_\eta[\,\tau_{\mathcal{A}} < \tau_{\mathcal{B}}\,]$$

$$\tau_{\mathcal{A}} \;=\; \inf\left\{\, t > 0 : \xi(t) \in \mathcal{A} \,\right\}$$

$$\operatorname{cap}(\mathcal{A},\, \mathcal{B}) \;=\; \mathfrak{D}(h_{\mathcal{A},\mathcal{B}})$$

$$\langle\, f \,,\, \mathfrak{L}\, g \,\rangle \;=\; \langle\, \mathfrak{L}\, f \,,\, g \,\rangle \quad \Longrightarrow \quad \operatorname{cap}(\mathcal{A},\, \mathcal{B}) \;=\; \inf_F \mathfrak{D}(F)$$

Non-reversible dynamics

$$\langle f \,,\, \mathcal{L}\, g \,\rangle \;\neq\; \langle \,\mathcal{L}\, f \,,\, g \,\rangle$$

$$\mathcal{D}(h_{\mathcal{A},\mathcal{B}}) \;\neq\; \inf_F \mathcal{D}(F) \;=\; \mathcal{D}(h^s_{\mathcal{A},\mathcal{B}}) \qquad \mathcal{L}^s h^s_{\mathcal{A},\mathcal{B}} \;=\; (1/2)(\mathcal{L} + \mathcal{L}^*) h^s_{\mathcal{A},\mathcal{B}} = 0$$

$$\mathrm{cap}(A,B) \;=\; \inf_f \,\sup_g \big\{\, 2\,\langle\, f \,,\, \mathcal{L}\, g\,\rangle \;-\; \mathcal{D}(g)\,\big\} \qquad f\chi_{A\cup B} = \chi_A \quad g \in \mathfrak{C}_{A,B}$$

$$\mathrm{cap}(A,B) \;=\; \inf_{f\in\mathfrak{C}_{1,0}(A,B)} \;\inf_{\phi\in\mathfrak{F}_0(A,B)} \|\Phi_f - \phi\|^2 \;.$$

$$\frac{1}{\mathrm{cap}(A,B)} \;=\; \inf_{\psi\in\mathfrak{F}_1(A,B)} \;\inf_{g\in\mathfrak{C}_{0,0}(A,B)} \|\Phi_g - \psi\|^2 \;.$$

- Lower and upper bounds

Doyle, Pinsky, Gaudillière, L., Slowik, Mariani, Seo, Probab. Surveys 2019

Resolvent approach to Metastability

Framework

$$E_N - \text{valued} \ \ \eta_N(t) \qquad \mathcal{E}_N = \bigcup_{x \in S} \mathcal{E}_N^x \quad \Delta_N = E_N \setminus \mathcal{E}_N$$

$$\eta^{\mathcal{E}}(t) \ \ \text{trace of } \eta(t) \text{ on } \mathcal{E}_N$$

$$R^{\mathcal{E}}(\eta, \xi) = \lambda(\eta)\, \mathbb{P}_\eta\big[\, \tau_F^+ = \tau_\xi \,\big]$$

$$\tau_A \ := \ \inf\{t \geq 0 : \eta_N(t) \in A\} \, , \quad \tau_A^+ \ = \ \inf\{t \geq \sigma_1 : \eta_N(t) \in A\}$$

$$\sigma_1 = \inf\{t \geq 0 : \eta_N(t) \neq \eta_N(0)\}$$

$$\Phi_N : \mathcal{E}_N \to S \qquad X_N(t) = \Phi_N(\eta^{\mathcal{E}}(t))$$

Mean jump rates

μ_N stationary state

$$r_N(x,y) \;=\; \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta\in\mathcal{E}_N^x} \mu_N(\eta) \sum_{\xi\in\mathcal{E}_N^y} R^{\mathcal{E}}(\eta,\xi)$$

$$r_N(x,y) \;=\; \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta\in\mathcal{E}_N^x} \mu_N(\eta)\, \lambda_N(\eta)\, \mathbb{P}_\eta^N\big[\, \tau_{\mathcal{E}_N^y} \,<\, \tau^+_{\cup_{z\neq y}\mathcal{E}_N^z} \,\big]$$

Reversible case:

$$2\,\mu_N(\mathcal{E}^x)\; r_N(x,y) \;=\; \mathrm{cap}_N\big(\, \mathcal{E}^x, \cup_{z\neq x}\mathcal{E}^z \,\big) +\; \mathrm{cap}_N\big(\, \mathcal{E}^y, \cup_{z\neq y}\mathcal{E}^z \,\big)$$

$$-\;\mathrm{cap}_N\big(\, \mathcal{E}^x \cup \mathcal{E}^y \,,\, \cup_{z\neq x,y}\mathcal{E}^z \,\big)$$

Resolvent condition $\mathfrak{R}$

$$G_N(\eta) \;=\; \sum_{x\in S} [\,(\,\lambda - L\,)\,f\,](x)\,\chi_{\mathcal{E}_N^x}(\eta)$$

$$(\,\lambda \,-\, \theta_N\,\mathfrak{L}_N\,)\,F_N \;=\; G_N$$

$$\lim_{N\to\infty}\,\max_{x\in S}\,\sup_{\eta\in\mathcal{E}_N^x}\,\Big|\,F_N(\eta)\,-\,f(x)\,\Big| \;=\; 0$$

Condition $\mathfrak{R}_{\rm cte}$: $\quad \forall\,(\lambda, f) \qquad \lim_{N\to\infty}\,\max_{x\in S}\,\max_{\eta,\zeta\in\mathcal{E}_N^x}\,|\,F_N(\eta)\,-\,F_N(\zeta)\,| \;=\; 0$

Condition $\mathfrak{R}_{\rm lim}$: $\quad f_N(x) \;=\; \dfrac{1}{\mu_N(\mathcal{E}_N^x)}\sum_{\eta\in\mathcal{E}_N^x} F_N(\eta)\,\mu_N(\eta) \;\longrightarrow\; f(x)$

Dynamics which visit points

Beltrán, L. (10-12)

Condition (H0). For all $x \neq y$, $r(x,y) \;=\; \lim_{N\to\infty} \theta_N \, r_N(x,y)$

Condition (H1). $\exists\ \xi_N^x \quad \lim_{N\to\infty} \max_{\eta\in\mathcal{E}_N^x} \dfrac{\mathrm{cap}_N(\mathcal{E}_N^x, \cup_{z\neq x}\mathcal{E}_N^z)}{\mathrm{cap}_N(\xi_N^x, \eta)} \;=\; 0$

L. Marcondes, Seo (21)

Theorem:

$(\mathrm{H0}),\ (\mathrm{H1}) \Rightarrow\ \mathfrak{R}_{\mathrm{cte}}$

$(\mathrm{H0}),\ (\mathrm{H1}),\ \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^y)} \to 0 \Rightarrow\ \mathfrak{R}_{\mathrm{lim}}$

Mixing conditions

Condition $\mathfrak{M}_1$: $\exists\, \zeta^x \in \mathcal{E}_N^x \quad \lim_{N\to\infty} \max_{\eta \in \mathcal{E}_N^y} \mathbf{P}_\eta^N \big[\, \tau_{\zeta_N^y} \ge s\, \theta_N \,\big] \;=\; 0$

Condition $\mathfrak{M}_2$: $\mathcal{E}_N^x \subset \mathcal{V}_N^x$

$$\lim_{N\to\infty} \sup_{\eta \in \mathcal{E}_N^x} \mathbf{P}_\eta^N \big[\, \tau_{(\mathcal{V}_N^x)^c} \le \mathbf{h}_N \,\big] \;=\; 0 \qquad t^x_{\rm mix}(\epsilon) \;\le\; \mathbf{h}_N$$

L., Marcondes, Seo ('21)

Proposition: (H0), $\mathfrak{M}_1 \;\Rightarrow \mathfrak{R}_{\rm cte}$ and $\mathfrak{R}_{\rm lim}$ if $\frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^y)} \to 0$

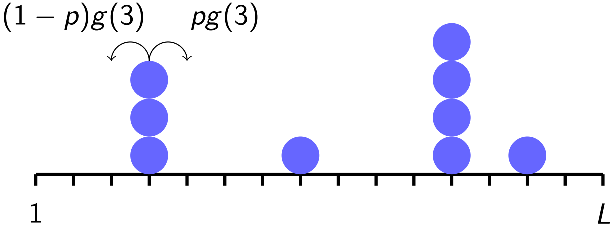
Proposition: (H0), $\mathfrak{M}_2 \;\Rightarrow \mathfrak{R}_{\rm cte}$ and $\mathfrak{R}_{\rm lim}$ if $\frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^y)} \to 0$

Proposition: $\mathfrak{M}_2 \;\Rightarrow \mathfrak{D}$ if $\mu(\Delta_N)/\mu_N(\mathcal{E}_N^x) \to 0$ for all x

Condensing zero-range dynamics

The model

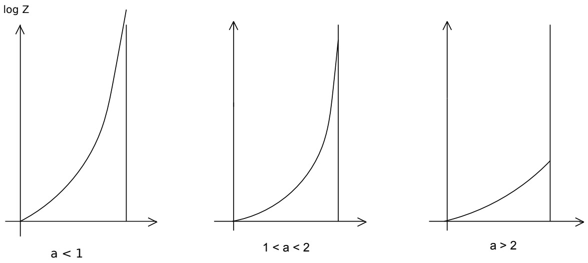
- $\mathbb{T}_L = \{1, \ldots, L\}$ state space $\mathbb{N}^{\mathbb{T}_L}$
- configurations $\eta = \{\eta_x : x \in \mathbb{T}_L\}$
- $g : \mathbb{N} \to \mathbb{R}_+$ $g(0) = 0$ $g(k) > 0$
- $1/2 \leq p \leq 1$
- $x \to x + 1$ at rate $pg(\eta_x)$ $x \to x - 1$ at rate $(1 - p)g(\eta_x)$



- $g(k)/k$ $g(k) = k$ $g(k) = 1$

Condensation

- $g(1) = 1 \quad g(k) = \left(\frac{k}{k-1}\right)^{\alpha} \quad k \geq 2 \quad \alpha > 0$
- Partition function: $Z(\varphi) = 1 + \sum_{k \geq 1} \frac{\varphi^k}{k^{\alpha}} \quad 0 \leq \varphi < 1$
- $\nu_{\varphi}\{\eta : \eta_x = k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{k^{\alpha}}$
- $\rho(\varphi) = E_{\nu_{\varphi}}[\eta_0] \qquad \rho(\varphi) = \varphi \frac{d}{d\varphi} \log Z(\varphi)$



Metastability

- Fix $1 \ll \ell_N \ll N$
- $\mathcal{E}_N^x = \{\eta : \eta_x \geq N - \ell_N\} \quad x \in S = \{1, \ldots, L\}$
- $\mathcal{E}_N^x \cap \mathcal{E}_N^y = \varnothing \quad \mathcal{E}_N = \bigcup_{x \in S} \mathcal{E}_N^x \quad E_N = \mathcal{E}_N \cup \Delta_N$
- $\mu_{L,N}(\mathcal{E}_N^x) \to 1/L \quad \mu_{L,N}(\mathcal{E}_N) \to 1$
- $\alpha > 1$ Beltrán, L. ('12), L. ('14), Seo ('19)
  - $\theta_N = N^{1+\alpha} \quad r(x,y) \sim \mathrm{cap}(x,y)$
  - (H0), (H1), ($\mathfrak{D}$)
- $\alpha = 1$ L., Marcondes Seo '20
  - $\theta_N = N^{1+\alpha} \log N = N^2 \log N$
  - (H1) does not hold $\mathfrak{M}_1 \quad \mathfrak{M}_2$