Speed of convergence to the mean-field limit for a mutation-selection particle system

Josué Corujo Rodríguez
work in collaboration with Bertrand Cloez (INRAE, Montpellier)

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Cornell University

Mathematics > Probability

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Uniform in time propagation of chaos for a Moran model

Bertrand Cloez, Josué Corujo

Fleming – Viot particle system

- **Problem:** estimate ν, the QSD of the absorbing Markov chain generated by $Q - \kappa$, where
  - $Q = (\mu_{x,y})_{x,y \in E}$ generator of a irreducible Markov chain
  - κ killing rate

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  $\eta = (\eta(1), \ldots, \eta(k), \ldots)$, s.t. $\eta(k) =$ number of particles of type k, and $|\eta| = N$

$$(\mathcal{Q}_N f)(\eta) = \sum_{x,y} \eta(x) \left(\mu_{x,y} + \kappa(x) \frac{\eta(y)}{N} \right) [f(\eta - \mathrm{e}_x + \mathrm{e}_y) - f(\eta)]$$

  The empirical measure $m(\eta_t^{(N)})$ approximates $\mathbb{P}[X_t \in \cdot \mid X_t \neq \partial]$ when $N \to \infty$

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Question

➥ Is this convergence uniform in time?

Moran-type particle system

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or even more general

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Some questions

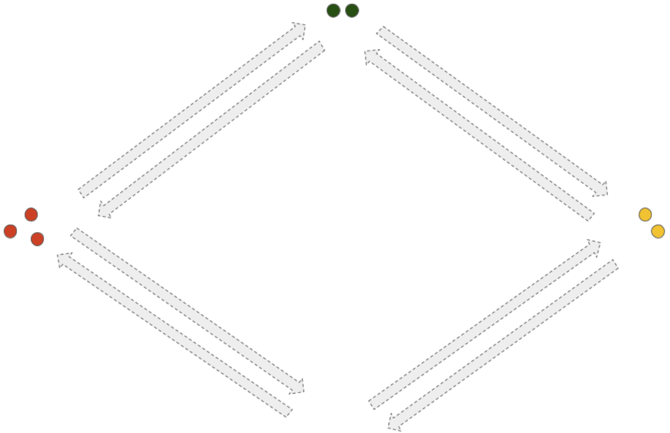
- Are these Moran-type particle processes also related to QSDs?
- What is the best algorithm ?

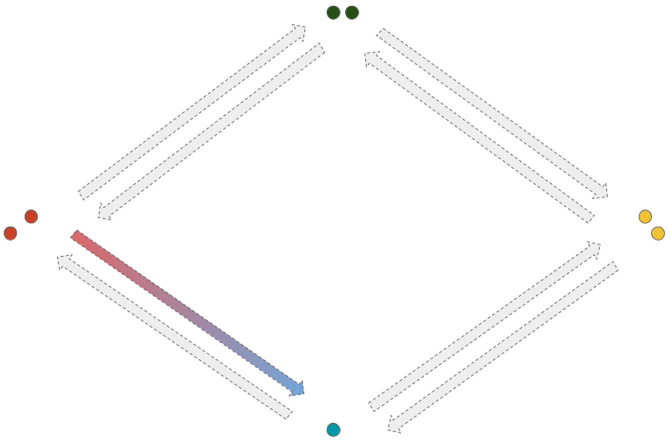
Multi-allelic Moran model

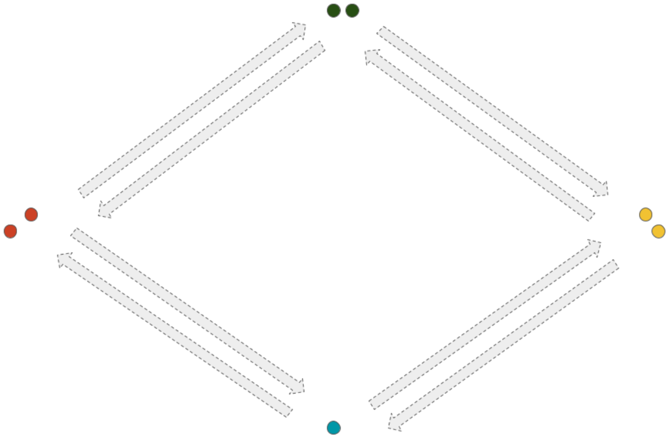
- set of possible allelic types: E (countable)
- number of individuals in the population: N
- state space of the process:

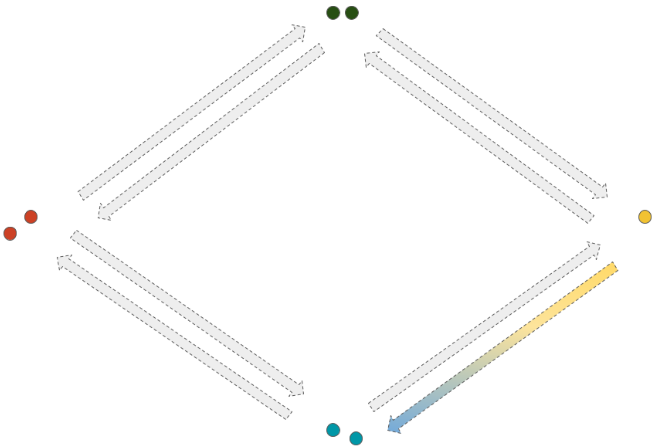
$$\mathcal{E}_N := \{\eta \in \mathbb{N}_0^E : \eta(1) + \cdots + \underbrace{\eta(k)}_{\substack{\text{nb. of indiv.}\\ \text{of type } k}} + \cdots = N\}$$

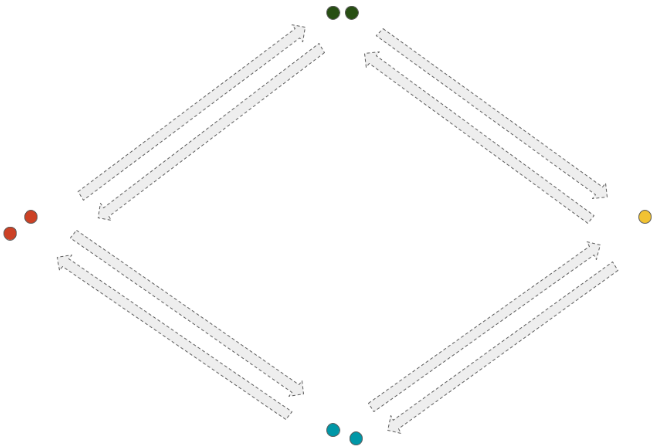
- state of the process at time t: $\eta_t^{(N)}$
- Interactions:
  - **mutation**: each individual mutates independently of the others according to an **irreducible** Markov chain
  - **reproduction**: one indiv. dies and another *randomly chosen* is duplicated (Moran type)

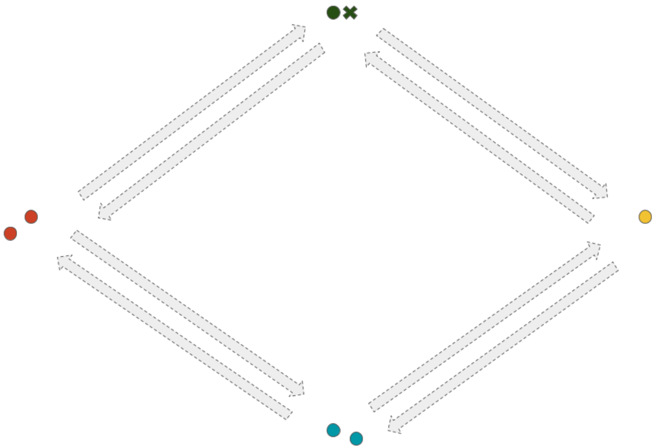


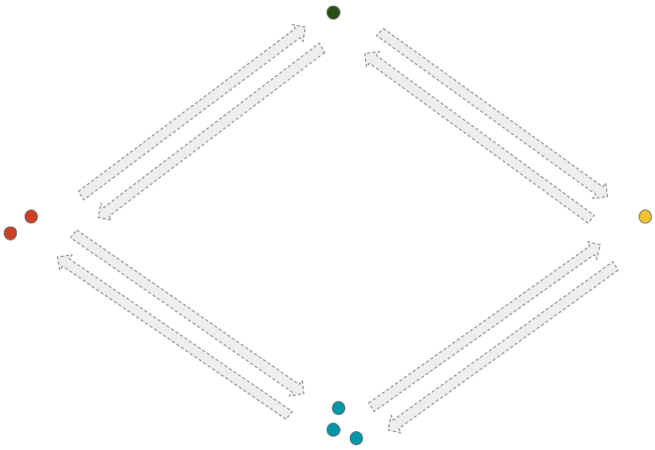


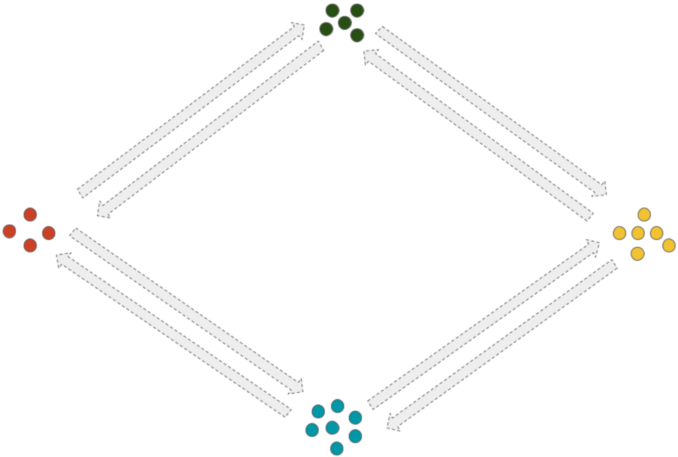












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- Induced empirical distribution: $m(\eta) := \sum_{x \in E} \frac{\eta(x)}{N} \delta_x$
- Generator:

$$\mathcal{Q}_N[\eta, \eta - \mathrm{e}_x + \mathrm{e}_y] = \eta(x) \left(\underbrace{q_{x,y}}_{\textbf{mutation}} + \underbrace{V_{m(\eta)}(x,y)}_{\textbf{reproduc. rate}} \underbrace{\frac{\eta(y)}{N}}_{\textbf{indiv. to reproduce is of type } y} \right)$$

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- selection at death: $V_\mu(x,y) = V^{\mathrm{d}}(x)$, **Fleming – Viot particle systems**
- additive selection: $V_\mu(x,y) = V^{\mathrm{d}}_\mu(x) + V^{\mathrm{b}}_\mu(y)$
  $\rightsquigarrow$ to favor the indiv. with relatively high values of $\Lambda = V^{\mathrm{b}}_\mu - V^{\mathrm{d}}_\mu$

Main problems and motivations

- Study the following convergences (existence, speed, ...)

$$
\begin{array}{ccc}
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{\scriptstyle N}\downarrow & & \downarrow{\scriptstyle N} \\
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- Particle systems *as a goal or as a tool*:
  - limit behavior of a given particle system
  - particle systems as a tool for approximating a quasi-stationary distribution

Outline

- **Additive reproduction rates**
- Propagation of chaos
- Asymptotic normality

Link to Feynman–Kac formulae and QSDs

(AD) Additive selection

$$V_\mu(x,y) = V_\mu^{\mathrm{d}}(x) + V_\mu^{\mathrm{b}}(y) + V_\mu^{\mathrm{s}}(x,y), \text{ such that}$$

$V_\mu^{\mathrm{b}} - V_\mu^{\mathrm{d}} = \Lambda + C_\mu$, where Λ is bounded and does not depend on μ

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Main questions

- What is the speed of convergence?
- What particle system is the best to approximate a given absorbing Markov chain?

Outline

- ✓ Additive reproduction rates
- ■ **Uniform in time propagation of chaos**
- ❑ Asymptotic normality

Convergence of the empirical measure

Hypothesis (IC) Initial condition (chaos or LLN)

$$\sup_{\|\phi\|\leq 1} \mathbb{E}[|m(\eta_0^{(N)})(\phi) - \mu_0(\phi)|^p]^{1/p} \leq \frac{C}{\sqrt{N}}$$

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$$\|\mu_t - \mu_\infty\|_{\mathrm{TV}} \leq C\mathrm{e}^{-\gamma t}, \text{ for all } \mu_0 \in \mathcal{M}_1(E) \text{ and } t \geq 0$$

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Theorem (Unif. in time propagation of chaos or LLN)

Assume that **(AD)**, **(IC)** and **(UC)** are verified. Then, for every $p \geq 1$,

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Idea of the proof: (Similar to Rousset (2006) for reversible diffusions)

- Base case: if $\Phi_t(\nu) := \text{Law}_\nu(X_t \mid t < \tau_\partial)$ then

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- Induction: martingale problem associated to $\mathcal{Q}_N$ on functions $\eta \mapsto m(\eta)(\phi)$ and Hölder inequality:
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- Thus

$$I_p(N) \leq \frac{C}{(N^{p/2})^{\min\{2\epsilon,1\}}} \implies I_p(N) \leq \frac{C}{N^{p/2}} \quad \blacksquare$$

Corollary

Assume that **(AD)**, **(IC)** and **(UC)** are verified. Then,

- Almost sure convergence:

$$m\left(\eta_T^{(N)}\right)(\phi) \xrightarrow[N\to\infty]{\text{a.s.}} \mu_T(\phi)$$

- Convergence of the mean empirical measure:

$$\sup_{t\geq 0}\left\|\bar{m}\left(\eta_t^{(N)}\right)-\mu_t\right\|_{\mathrm{TV}} \leq \frac{C}{N}, \text{ where } \bar{m}(\eta_t^{(N)}) := \sum_{x\in E}\mathbb{E}\left[\frac{\eta_t^{(N)}(x)}{N}\right]\delta_x$$

- Moreover, if the initial distribution of the N particles is exchangeable, then

$$\sup_{t\geq 0}\left\|\mathrm{Law}(\xi_t^{(i)})-\mu_t\right\|_{\mathrm{TV}} \leq \frac{C}{N}$$

Outline

- ✓ Additive reproduction rates
- ✓ Uniform in time propagation of chaos
- ■ **Asymptotic normality**

Asymptotic normality

- Our result is a Law of Large Numbers

$$\sup_{\|\phi\|\leq 1} \sup_{t\geq 0} \mathbb{E}\left[\left|m\left(\eta_t^{(N)}\right)(\phi) - \mu_t(\phi)\right|^p\right]^{1/p} \leq \frac{C_p}{\sqrt{N}}$$

- **What about a Central Limit Theorem?**

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- **What about a Central Limit Theorem?**

Hypothesis (IC') Asymptotic normality for initial empirical distribution

$$\sqrt{N}\left(m(\eta_0^{(N)})(\phi) - \mu_0(\phi)\right) \xrightarrow[N\to\infty]{\text{Law}} \mathcal{N}\left(0, \mu_0(\phi^2)\right)$$

Theorem (Asymptotic normality)

Suppose that Assumptions **(AD)**, **(IC)**, **(IC')** and **(UC)** are verified.
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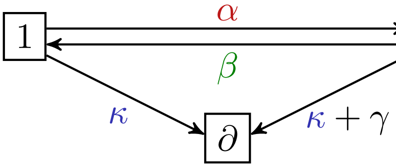
Reduction of variance

Take $(\eta_t^\star)_{t\geq 0}$ with selection rates $V_\mu - V_\mu^{\mathrm{s}}$. Then,

$$\lim_{N\to\infty} N\mathbb{E}\left[\left(m(\eta_T^\star)(\phi) - \mu_T(\phi)\right)^2\right] \leq \lim_{N\to\infty} N\mathbb{E}\left[\left(m(\eta_T)(\phi) - \mu_T(\phi)\right)^2\right]$$

Simplest example: two-state absorbing Markov chain

- **Problem:** estimate $\nu_\kappa(1)$: ν_κ the QSD of the absorbing Markov chain



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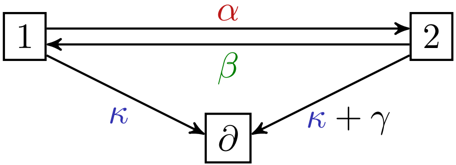
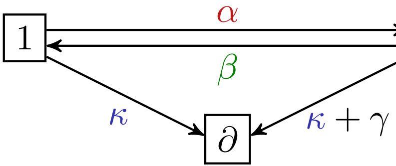


- **Fact:** $\mathbb{P}[X_t \in \cdot \mid \tau_\partial > t]$ is independent of $\kappa \implies \nu_\kappa = \nu_0$

Simplest example: two-state absorbing Markov chain

- **Problem:** estimate $\nu_\kappa(1)$: ν_κ the QSD of the absorbing Markov chain



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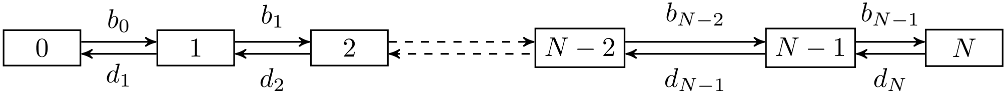

with rates $d_n = n\left(\alpha + \kappa \frac{N-n}{N}\right)$ and $b_n = (N-n)\left(\beta + (\kappa+\gamma)\frac{n}{N}\right)$

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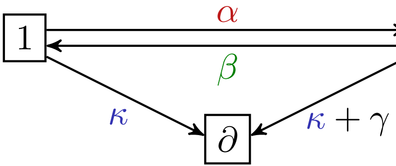

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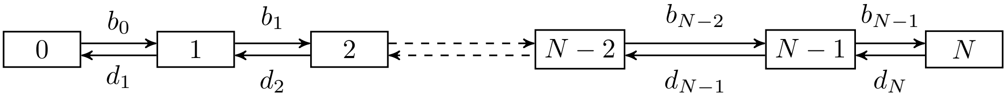

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- **Corollary:** $\mathcal{Z}^{(0)}$ estimates ν_κ with smaller asymptotic squared error

Merci beaucoup

pour votre attention !

Speed of convergence to the mean-field limit for a mutation-selection particle system

Josué Corujo Rodríguez
work in collaboration with Bertrand Cloez (INRAE, Montpellier)

Workshop de l'ANR QuAMProcs · March 8th, 2022