On a structure-preserving numerical method for fractional Fokker-Planck

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Quantitative Analysis of Metastable Processes
March 8, 2022

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The fractional kinetic Fokker-Planck equation

The Lévy-Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f \; = \; \nabla_v \cdot (vf) - (-\Delta_v)^{\alpha/2} f =: L_\alpha f \,,$$

$t \geqslant 0$, $x \in \mathbb{T}^d$ and $v \in \mathbb{R}^d$; $\alpha \in (0, 2)$.

For any nice function $g : \mathbb{R}^d \to \mathbb{R}$,

$$\mathcal{F}((-\Delta_v)^{\alpha/2} g)(\xi) \; = \; |\xi|^\alpha \mathcal{F}(g)(\xi)$$

where $\mathcal{F}(\cdot)$ the Fourier transform.

Another equivalent definition:

$$(-\Delta_v)^{\alpha/2} g(v) \; = \; C_{d,\alpha} \, \mathrm{P.V.} \int_{\mathbb{R}^d} \frac{g(v) - g(w)}{|v - w|^{d+\alpha}} \, \mathrm{d}w \,,$$

where $\mathrm{P.V.}$ the principal value.

Hypocoercivity

The local equilibrium: μ_α, a probability distribution such that $L_\alpha \mu_\alpha = 0$. It decays as $|v|^{-\alpha-d}$ when $|v| \to \infty$.

Exponential return to equilibrium: there exists an appropriate functional space X and constants $\lambda > 0$, $C \geqslant 1$ such that

$$\|f(t) - \left\langle f^0 \right\rangle \mu_\alpha\|_X \;\leqslant\; C \,\|f^0 - \left\langle f^0 \right\rangle \mu_\alpha\|_X \, e^{-\lambda t}$$

with $\left\langle f^0 \right\rangle := \int\int_{\mathbb{T}^d \times \mathbb{R}^d} f^0 \mathrm{d}x\mathrm{d}v$.

The homogeneous case: Gentil-Imbert (2008), Tristani (2014).

Our approach: The H^1 method. For the L^2 method, see Bouin et al (2019).

(Hypo)coercive schemes

Aim: **A numerical approach**.

The classical case: Dujardin et al (2020) [H^1 method], Bessemoulin et al (2020) [L^2 method].

Goal: Design of a **consistent**, **stable** and **structure preserving** numerical method for $d = 1$.

The continuous case

The Sobolev space $H^1_{x,v}(\nu)$ is associated with the norm

$$\|g\|^2_{H^1_{x,v}(\nu)} = \|g\|^2_{L^2_{x,v}(\nu)} + \|\nabla_x g\|^2_{L^2_{x,v}(\nu)} + \|\nabla_v g\|^2_{L^2_{x,v}(\nu)} .$$

Theorem (Ayi, Herda, Hivert, Tristani (2020))

Let f solve **the kinetic Lévy-Fokker-Planck equation** *with initial data $f^{in} \in H^1_{x,v}(\mu_\alpha^{-1})$. Then, for all $t \geqslant 0$ one has*

$$\|f(t) - \left\langle f^{in} \right\rangle \mu_\alpha\|_{H^1_{x,v}(\mu_\alpha^{-1})} \leqslant C\,\|f^{in} - \left\langle f^{in} \right\rangle \mu_\alpha\|_{H^1_{x,v}(\mu_\alpha^{-1})}\, e^{-\lambda t}$$

for some constant $C \geqslant 1$ and $\lambda > 0$ depending only on d and α.

Idea: Carry out our computations as **simple** as possible to **adapt our analysis** to a **discrete framework**.

Difficulty: comparison with the non fractional case $\Rightarrow$ lack of symmetry of our operator in $L^2_v(\mu_\alpha^{-1})$.

Proposition

One has the decomposition

$$-\langle L_\alpha f,\, g\rangle_{L^2_v(\mu_\alpha^{-1})} \;=\; \mathcal{S}_v(f,g) \,+\, \mathcal{A}_v(f,g)\,,$$

where $\mathcal{S}_v$ *and* $\mathcal{A}_v$ *are bilinear forms that are respectively symmetric and skew-symmetric and defined by*

$$\mathcal{S}_v(f,g) = \frac{C_{d,\alpha}}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\left[(f\mu_\alpha^{-1})(v) - (f\mu_\alpha^{-1})(w)\right]\left[(g\mu_\alpha^{-1})(v) - (g\mu_\alpha^{-1})(w)\right]}{|v-w|^{d+\alpha}} \mu_\alpha(v)\mathrm{d}w\mathrm{d}v$$

and

$$\mathcal{A}_v(f,g) \;=\; \frac{C_{d,\alpha}}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f\mu_\alpha^{-1})(w)(g\mu_\alpha^{-1})(v) - (f\mu_\alpha^{-1})(v)(g\mu_\alpha^{-1})(w)}{|v-w|^{d+\alpha}} \mu_\alpha(v)\,\mathrm{d}w\,\mathrm{d}v$$
$$+\,\frac{1}{2}\int_{\mathbb{R}^d} \left(f\,v\cdot\nabla_v(g\mu_\alpha^{-1}) \,-\, g\,v\cdot\nabla_v(f\mu_\alpha^{-1})\right)\mathrm{d}v.$$

Some useful inequalities

The **nullspace** of L_α is exactly given by $\mathbb{R}\mu_\alpha$. The **orthogonal projection** Π onto the nullspace of L_α is given by

$$(\Pi g)(v) \;=\; \left(\int_{\mathbb{R}^d} g(w)\,\mathrm{d}w\right)\,\mu_\alpha(v)\,.$$

Lemma (Chafai (2004), Gentil-Imbert (2008), Wang (2014) ...)

There is a constant $C_P \equiv C_P(\alpha, d) > 0$ such that for all $f \in D(L_\alpha)$,

$$\|f - \Pi f\|^2_{L^2_v(\mu_\alpha^{-1})} \;\leqslant\; C_P\,\mathcal{S}_v(f,f)\,.$$

The dissipation $\mathcal{S}_v(f,f)$ also provides **some fractional Sobolev regularity**.

The fractional Sobolev space H^s_v with $s \in (0,1)$:

$$\|\cdot\|^2_{H^s_v} = \|\cdot\|^2_{L^2_v} + \|\cdot\|^2_{\dot{H}^s_v}$$

where the homogeneous Sobolev norm is given by

$$\begin{aligned} \|g\|^2_{\dot{H}^s_v} &:= \|(-\Delta)^{s/2} g\|^2_{L^2_v} \\ &= \widetilde{C}_{d,s} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(v) - f(w)|^2 |v-w|^{-(d+2s)} \,\mathrm{d}w\,\mathrm{d}v \end{aligned}$$

for some positive constant $\widetilde{C}_{d,s}$.

Proposition

There is $C_F \equiv C_F(\alpha, d)$ such that for all $f \in D(L_\alpha)$,

$$\|(f - \Pi f)\mu_\alpha^{-1/2}\|^2_{H_v^{\alpha/2}} \leqslant C_F\, \mathcal{S}_v(f,f)\,.$$

Proposition

For all $\varepsilon > 0$, there is $K(\varepsilon) \equiv K(\varepsilon, \alpha, d) > 0$ such that

$$\|\nabla_v f\|^2_{L^2_v(\mu_\alpha^{-1})} \leqslant K(\varepsilon) \left(\mathcal{S}_v(f,f) + \|\Pi f\|^2_{L^2_v(\mu_\alpha^{-1})} \right) + \varepsilon\, C_F\, \mathcal{S}_v(\nabla_v f, \nabla_v f).$$

Proof of the theorem

Up to changing f^{in} by $f^{\text{in}} - \langle f^{\text{in}} \rangle \mu_\alpha$, we assume that $\iint_{\mathbb{T}^d \times \mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0$ at $t = 0$, so that by conservation it also holds for all time $t > 0$.

Functional on **the weighted Sobolev space** $H^1_{x,v}(\mu_\alpha^{-1})$ defined by

$$\mathcal{H}(f,f) = \|f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} + a\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} + b\,\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} + 2\,c\,\langle \nabla_x f, \nabla_v f \rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\,.$$

For well-chosen positive constants $a, b, c > 0$, it is equivalent to **the weighted Sobolev norm** $\|f\|^2_{H^1_{x,v}(\mu_\alpha^{-1})}$.

The commutators:

$$\begin{array}{ll} [\nabla_x, v\cdot\nabla_x] = 0, & [\nabla_x, L_\alpha] = 0, \\ [\nabla_v, v\cdot\nabla_x] = \nabla_x, & [\nabla_v, L_\alpha] = \nabla_v. \end{array}$$

The notation $\mathcal{S}_{x,v}$ denotes the integral of $\mathcal{S}_v$ in the x variable.

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= \langle L_\alpha f, f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle v\cdot\nabla_x f, f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\ &= -\mathcal{S}_{x,v}(f,f)\end{aligned}$$

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$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= \langle \nabla_x L_\alpha f, \nabla_x f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_x (v\cdot\nabla_x f), \nabla_x f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\
&= \langle L_\alpha \nabla_x f, \nabla_x f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle (v\cdot\nabla_x)\nabla_x f, \nabla_x f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\
&= -\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f)
\end{aligned}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} = -\mathcal{S}_{x,v}(f,f)\,,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} = -\mathcal{S}_{x,v}(\nabla_x f,\nabla_x f)$$

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= \langle \nabla_v L_\alpha f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_v(v\cdot\nabla_x f), \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\
&= \langle L_\alpha \nabla_v f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle (v\cdot\nabla_x)\nabla_v f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\
&\quad + \|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\
&= -\mathcal{S}_{x,v}(\nabla_v f,\nabla_v f) + \|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}
\end{aligned}$$

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$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} = -\mathcal{S}_{x,v}(\nabla_x f,\nabla_x f)\,,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} = -\mathcal{S}_{x,v}(\nabla_v f,\nabla_v f) + \|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} &= \langle \nabla_x L_\alpha f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} + \langle \nabla_x f, \nabla_v L_\alpha f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\ &\quad - \langle \nabla_x (v\cdot\nabla_x f), \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle \nabla_x, \nabla_v (v\cdot\nabla_x f) f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\ &= -\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - \mathcal{S}_{x,v}(\nabla_x f,\nabla_v f) + \langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\end{aligned}$$

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= -\mathcal{S}_{x,v}(f,f)\,,\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= -\mathcal{S}_{x,v}(\nabla_x f,\nabla_x f)\,,\\
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} &= -\mathcal{S}_{x,v}(\nabla_v f,\nabla_v f) + \|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - \langle\nabla_x f,\nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\,,\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\nabla_x f,\nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} &= -\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - 2\,\mathcal{S}_{x,v}(\nabla_x f,\nabla_v f) + \langle\nabla_x f,\nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\,.
\end{aligned}
$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} = -\mathcal{S}_{x,v}(f,f)\,,$$
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$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} = -\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - 2\,\mathcal{S}_{x,v}(\nabla_x f,\nabla_v f) + \langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\,.$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f,f) \;=\; & -\mathcal{S}_{x,v}(f,f) - a\mathcal{S}_{x,v}(\nabla_x f,\nabla_x f) - b\mathcal{S}_{x,v}(\nabla_v f,\nabla_v f) - c\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} \\ & + b\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} - b\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})} \\ & - 2c\,\mathcal{S}_{x,v}(\nabla_x f,\nabla_v f) + c\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\,.\end{aligned}$$

First four terms: dissipation terms, last four terms: remainder terms.

A direct consequence of the **Cauchy-Schwarz inequality** is

$$\mathcal{S}_v(f,g) \leqslant \mathcal{S}_v(f,f)^{1/2}\,\mathcal{S}_v(g,g)^{1/2}.$$

By integrating in x and using **Young's inequality**, we get

$$|2c\,\mathcal{S}_{x,v}(\nabla_x f, \nabla_v f)| \leqslant \frac{2\,c^2}{b}\,\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f) + \frac{b}{2}\,\mathcal{S}_{x,v}(\nabla_v f, \nabla_v f).$$

Then since $\int \nabla_v f \,\mathrm{d}v = 0$, one has

$$\begin{aligned} b\left|\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\right| &= b\left|\langle \nabla_x f - \Pi(\nabla_x f), \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\right| \\ &\leqslant b\,C_P^{1/2}\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f)^{1/2}\|\nabla_v f\|_{L^2_{x,v}(\mu_\alpha^{-1})} \\ &\leqslant \frac{b\,C_P}{2}\,\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f) + \frac{b}{2}\,\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}. \end{aligned}$$

Similarly

$$c\left|\langle \nabla_x f, \nabla_v f\rangle_{L^2_{x,v}(\mu_\alpha^{-1})}\right| \leqslant \frac{c^2\,C_P}{2b}\,\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f) + \frac{b}{2}\,\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}.$$

For the last remainder term we integrate our **useful estimate** in x,

$$\|\nabla_v f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} \leqslant K(\varepsilon)\left(\mathcal{S}_{x,v}(f,f) + \|\Pi f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}\right) + \varepsilon\, C_F\, \mathcal{S}_{x,v}(\nabla_v f, \nabla_v f)\,.$$

We can use the **Poincaré inequality on the torus** (since f is mean-free) and the **Jensen inequality** to get

$$\|\Pi f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})} \leqslant \widetilde{C}_P \|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}$$

where $\widetilde{C}_P \equiv \widetilde{C}_P(d)$ is the Poincaré constant of the d-dimensional torus.

Thus eventually, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f,f) + D(f,f) \leqslant 0\,,$$

where the **dissipation** is given by

$$\begin{aligned} D(f,f) \;=\; & (1-2\,b\,K(\varepsilon))\mathcal{S}_{x,v}(f,f) + \left(a - \frac{c^2}{b}\left(2 + \frac{C_P}{2}\right) - \frac{bC_P}{2}\right)\mathcal{S}_{x,v}(\nabla_x f, \nabla_x f) \\ & + \left(\frac{b}{2} - 2b\varepsilon C_F\right)\mathcal{S}_{x,v}(\nabla_v f, \nabla_v f) + \left(c - 2b\widetilde{C}_P K(\varepsilon)\right)\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}\,. \end{aligned}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(f,f)+D(f,f)\leqslant 0\,,$$

with

$$\begin{aligned} D(f,f) \;&=\; (1-2\,b\,K(\varepsilon))\mathcal{S}_{x,v}(f,f) \;+\; \left(a-\frac{c^2}{b}\left(2+\frac{C_P}{2}\right)-\frac{bC_P}{2}\right)\mathcal{S}_{x,v}(\nabla_x f,\nabla_x f) \\ &\quad + \left(\frac{b}{2}-2b\varepsilon C_F\right)\mathcal{S}_{x,v}(\nabla_v f,\nabla_v f)+\left(c-2b\widetilde{C}_P K(\varepsilon)\right)\|\nabla_x f\|^2_{L^2_{x,v}(\mu_\alpha^{-1})}\,. \end{aligned}$$

We choose consecutively ε, b, c and a such that $0<\varepsilon<1/(4C_F)$, $0<b<1/(2K(\varepsilon))$, $c>2b\widetilde{C}_P K(\varepsilon)$ and finally a large enough so that $a>c^2\left(2+C_P/2\right)/b+bC_P/2$. It yields that the **dissipation is non-negative** .

There is a constant $\lambda>0$ (depending on a,b,c,ε) such that $D(f,f)\geqslant\lambda\mathcal{H}(f,f)$. By a **Gronwall type argument** we have that

$$\mathcal{H}(f,f) \text{ decays exponentially to 0 when } t\to\infty.$$

Challenges of the discrete setting

Aim: Design of a **consistent**, **stable** and **structure preserving** numerical method for $d = 1$.

Preservation of the structure:

- conservation of mass;
- preservation of the heavy-tailed local equilibrium μ_α;
- preservation of coercivity properties in the homogeneous case;
- preservation of the hypocoercivity properties in the inhomogeneous case;
- approximation of the fractional Fokker-Planck operator L_α on the whole line with a discretization on a truncated domain;
- preservation of the asymptotics $\alpha \to 2^-$,
- preservation of non-negativity of solutions observed numerically.

Challenges of the discrete setting

Aim: Design of a **consistent**, **stable** and **structure preserving** numerical method for $d = 1$.

Result:

- Rigorous **coercivity and hypocoercivity properties** $\Rightarrow$ exponential stability of the discrete solution.
- Discrete functional analysis :
  - A **discrete** version of **nonlocal Poincaré inequalities**,
  - new interpolation and embedding inequalities involving **discrete fractional operators and norms**.

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Result:

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- Discrete functional analysis :
  - A **discrete** version of **nonlocal Poincaré inequalities**,
  - new interpolation and embedding inequalities involving **discrete fractional operators and norms**.

Difficulties: **No more Fourier analysis**, the **commutators** between discrete operators contain **remainder terms** which vanish when the mesh size goes to 0 but need to be dealt with in order to close estimates.

Presentation of the numerical method (unbounded velocity domain)

Discretization of $\mathbb{R}$: $(v_j = jh)_{j\in\mathbb{Z}}$ with $h > 0$.

For a velocity distribution $f : \mathbb{R} \to \mathbb{R}$,

$$f_j \approx f(v_j).$$

Slight abuse of notation: $f = (f_j)_{j\in\mathbb{Z}}$.

Discretization of the fractional Laplacian: $\Lambda_\alpha^h : \mathbb{R}^\mathbb{Z} \to \mathbb{R}^\mathbb{Z}$ such that

$$(\Lambda_\alpha^h f)_j \approx -(-\Delta)^{\alpha/2} f(v_j).$$

The Huang-Oberman method (2014): the discrete fractional Laplace operator is

$$(\Lambda_\alpha^h f)_j = \sum_{k=1}^{\infty} \beta_k^h \, (f_{j+k} + f_{j-k} - 2f_j)\, h = \sum_{k\in\mathbb{Z}} \beta_k^h \, (f_{j-k} - f_j)\, h \,,$$

Lemma

There exist positive constants b_α and B_α depending only on $\alpha \in (0,2)$ such that

$$\frac{b_\alpha}{|hk|^{1+\alpha}} \leqslant \beta_k^h \leqslant \frac{B_\alpha}{|hk|^{1+\alpha}}, \quad \forall k \in \mathbb{Z}\backslash\{0\}.$$

Lemma (Huang-Oberman (2014))

Conservation of mass:

$$\sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h u)_j \; = \; 0 \, .$$

Self-adjoint *in the space of square summable sequences:*

$$\sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h u)_j v_j \; = \; \sum_{j\in\mathbb{Z}} (\Lambda_\alpha^h v)_j u_j \, .$$

Consistency *with the usual centered finite difference* **approximation of the Laplacian***:*

$$\lim_{\alpha\to 2^-} (\Lambda_\alpha^h u)_j \; = \; \frac{u_{j+1} + u_{j-1} - 2u_j}{h^2} \, ,$$

for all $j \in \mathbb{Z}$*.*
Consistency *at order* $3-\alpha$*. When* $h \to 0$*, one has for any* $u \in \mathcal{C}_b^4(\mathbb{R})$ *that*

$$\sup_{j\in\mathbb{Z}} \left| -(-\Delta)^{\alpha/2} u(hj) - (\Lambda_\alpha^h u)_j \right| \; \leqslant \; K_\alpha \, \|u\|_{\mathcal{C}_b^4(\mathbb{R})} \, h^{3-\alpha} \, ,$$

with K_α *a positive constant depending only on* α*.*

Discretization of the Lévy-Fokker-Planck operator: $L_\alpha^h = \Gamma_\alpha^h + \Lambda_\alpha^h$
where Γ_α^h discrete equivalent of $\partial_v(v\,\cdot)$.

Goal: define a **consistent approximation** that **preserves** exactly **the discrete equilibrium** $(M_j)_{j\in\mathbb{Z}}$ defined by

$$M_j = \mu_\alpha(v_j)\,.$$

Idea: Using that $L_\alpha\mu_\alpha = 0$ and that μ_α is symmetric, we get

$$\partial_v(v\,f) = \partial_v\left(v\,\mu_\alpha\,f/\mu_\alpha\right)$$

and

$$v\mu_\alpha(v) := \frac{1}{2}\int_{-v}^{v}(-\Delta_w)^{\alpha/2}\mu_\alpha(w)\mathrm{d}w\,,$$

The operator Γ_α^h is

$$(\Gamma_\alpha^h f)_j := \frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{h}$$

with the numerical flux defined by the centered approximation

$$\mathcal{F}_{j+\frac{1}{2}} := (VM)_{j+\frac{1}{2}}\left(\frac{f_j}{2M_j} + \frac{f_{j+1}}{2M_{j+1}}\right),$$

and

$$(VM)_{j+\frac{1}{2}} = -\,(VM)_{-j-\frac{1}{2}} := -\frac{1}{2}\sum_{k=-j}^{j}(\Lambda_\alpha^h M)_k\,h\,,\quad \text{for } j \geqslant 0\,.$$

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and

$$(VM)_{j+\frac{1}{2}} = -(VM)_{-j-\frac{1}{2}} := -\frac{1}{2} \sum_{k=-j}^{j} (\Lambda_\alpha^h M)_k \, h\,, \quad \text{for } j \geqslant 0\,.$$

For any odd $m \in \mathbb{Z}$,

$$(VM)_{j+\frac{m}{2}} \;=\; \frac{1}{4} \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} \mathrm{sgn}\left(k - (j + \tfrac{m}{2})\right) \beta_\ell^h (M_{k+\ell} + M_{k-\ell} - 2M_k)\, h^2\,,$$

Lemma (Basic properties)

The operator L_α^h satisfies the following properties.

i) **Mass conservation:** *for any suitably summable sequence u, one has*

$$\sum_{j\in\mathbb{Z}} (L_\alpha^h u)_j \, h \;=\; 0\,.$$

ii) **Preservation of local equilibrium:**

$$(L_\alpha^h M)_j \;=\; 0\,, \quad \forall j \in \mathbb{Z}\,.$$

iii) **Consistency:** *for any $u \in \mathcal{C}_b^4(\mathbb{R})$, one has that*

$$\sup_{j\in\mathbb{Z}} \left|(L_\alpha u)(hj) - (L_\alpha^h u)_j\right| \;\leqslant\; K_\alpha \, \|u\|_{\mathcal{C}_b^4(\mathbb{R})} \, h^{\min(3-\alpha,2)}\,,$$

for some $K_\alpha > 0$.

Proposition (Bilinear decomposition)

Given $(f_j)_{j\in\mathbb{Z}}$ and $(g_j)_{j\in\mathbb{Z}}$, we introduce $F_j = f_j/M_j$ and $G_j = g_j/M_j$ for any $j \in \mathbb{Z}$. One has the following decomposition

$$-\sum_{j\in\mathbb{Z}} (L^h_\alpha f)_j \, g_j \, M_j^{-1} h \;=\; \mathcal{S}^h_\alpha(f,g) \;+\; \mathcal{A}^h_\alpha(f,g)$$

where $\mathcal{S}^h_\alpha$ and $\mathcal{A}^h_\alpha$ are respectively symmetric and skew-symmetric bilinear forms defined by

$$\mathcal{S}^h_\alpha(f,g) = \frac{1}{2}\sum_{(j,k)\in\mathbb{Z}^2} \beta^h_k (F_j - F_{j+k})\,(G_j - G_{j+k})\, M_j\, h^2\,,$$

and

$$\mathcal{A}^h_\alpha(f,g) = -\frac{1}{2}\sum_{(j,k)\in\mathbb{Z}^2} \beta^h_k (F_j\, G_{j+k} - G_j\, F_{j+k})\, M_j\, h^2 - \frac{1}{2}\sum_{j\in\mathbb{Z}} (VM)_{j+\frac{1}{2}} (F_{j+1}\, G_j - F_j\, G_{j+1}).$$

Corollary

The **discrete Lévy-Fokker-Planck operator** L_α^h*, as an operator on* $\{(f_j)_j \mid \sum_j f_j^2 M_j^{-1} < +\infty\}$*, has the following properties:*

i) $\mathrm{Ker}(L_\alpha^h) = \mathrm{span}\{(M_j)_{j\in\mathbb{Z}}\}$

ii) $\mathrm{Im}(L_\alpha^h) \subset \{(g_j)_j \mid \sum_j g_j = 0\}$

Numerical schemes

The homogeneous case: For a time discretization $t_n = n\Delta t$ with time step $\Delta t > 0$, $f_j^n \approx f(t_n, v_j)$ is computed by solving the **implicit in time scheme**

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = (L_\alpha^h f)_j^{n+1}, \quad \forall (n, j) \in \mathbb{N} \times \mathbb{Z},$$

and starts at some given initial data $(f_j^0)_j$.

The inhomogeneous case: For any $(n, i, j) \in \mathbb{N} \times \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}$, $t_n = n\Delta t$, $x_i = i\Delta x$ and $v_j = j\Delta v$ where $\Delta x = N_x^{-1}$ with N_x an odd positive integer is **the space step** and $\Delta v > 0$ **the velocity step**.
$f_{ij}^n \approx f(t_n, x_i, v_j)$ is computed by solving the **implicit in time scheme**

$$\frac{f_{ij}^{n+1} - f_{ij}^n}{\Delta t} + (T^{\Delta x} f)_{ij}^{n+1} = (L_\alpha^{\Delta v} f)_{ij}^{n+1}, \quad \forall (n, i, j) \in \mathbb{N} \times \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z},$$

with given initial data $(f_{ij}^0)_{ij}$. The **discrete transport operator** writes

$$(T^{\Delta x} f)_{ij}^n = v_j \frac{f_{i+1,j}^n - f_{i-1,j}^n}{2\Delta x}.$$

Proposition

The scheme satisfies the following properties.

i) A solution $(f_{ij}^n)_{i,j,n}$ *is* **a stationary solution**, *i.e.* $f_{ij}^{n+1} = f_{ij}^n$ *for all* $n \geqslant 0$, $i \in \mathbb{Z}/N_x\mathbb{Z}$ *and* $j \in \mathbb{Z}$, *if and only if for some constant* $C \in \mathbb{R}$,

$$f_{ij}^n = CM_j = C\mu_\alpha(v_j), \quad \forall (n,i,j) \in \mathbb{N} \times \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}.$$

ii) **The total mass is preserved**, *namely for any suitably summable initial data* $(f_{ij}^0)_{ij}$

$$\sum_{i \in \mathbb{Z}/N_x\mathbb{Z}} \sum_{j \in \mathbb{Z}} f_{ij}^n \, \Delta v \, \Delta x = \sum_{i \in \mathbb{Z}/N_x\mathbb{Z}} \sum_{j \in \mathbb{Z}} f_{ij}^0 \, \Delta v \, \Delta x, \quad \forall n \in \mathbb{N}.$$

iii) The solution satisfies the following **global stability estimate**

$$\sum_{i \in \mathbb{Z}/N_x\mathbb{Z}} \sum_{j \in \mathbb{Z}} (f_{ij}^n)^2 M_j^{-1} \, \Delta v \, \Delta x \leqslant \sum_{i \in \mathbb{Z}/N_x\mathbb{Z}} \sum_{j \in \mathbb{Z}} (f_{ij}^0)^2 M_j^{-1} \, \Delta v \, \Delta x, \quad \forall n \in \mathbb{N}.$$

Discrete functional analysis

The weighted discrete Lebesgue space $\ell_h^2(\gamma)$: for a sequence $(g_j)_{j\in\mathbb{Z}}$, we define

$$\|g\|^2_{\ell_h^2(\gamma)} = \sum_{j\in\mathbb{Z}} g_j^2\,\gamma_j\,h\,.$$

Weighted discrete fractional Sobolev seminorms

$$|g|^2_{\dot{H}_h^s(\gamma)} = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{(g_j - g_{j+k})^2}{|hk|^{1+2s}}\,\gamma_j\,h^2\,, \quad s>0$$

and norms

$$\|g\|^2_{H_h^s(\gamma)} = \|g\|^2_{\ell_h^2(\gamma)} + |g|^2_{\dot{H}_h^s(\gamma)}\,, \quad \forall s\in(0,1)\,.$$

The **finite difference operators**:

$$(D_h^+ g)_j = \frac{g_{j+1}-g_j}{h},\ (D_h g)_j = \frac{g_{j+1}-g_{j-1}}{2h},\ ((D_h^2 g)_j = \frac{g_{j+2}+g_{j-2}-2g_j}{4h^2}$$

The **integration by part** formula: $\sum_{j\in\mathbb{Z}}(D_h g)_j f_j = -\sum_{j\in\mathbb{Z}} g_j (D_h f)_j\,.$

Weighted H_h^1-**Sobolev spaces** norm

$$\|g\|^2_{H_h^1(\gamma)} = \|g\|^2_{\ell_h^2(\gamma)} + \|D_h g\|^2_{\ell_h^2(\gamma)}\,.$$

Discrete non-local Poincaré inequalities

$A \lesssim B$ means there is a positive constant C which may depend on α, and other parameters, but never on the mesh size h such than $A \leqslant C\,B$.

Proposition

For any suitably summable sequence $f = (f_j)_j$, one has

$$\|f - \Pi_h f\|^2_{\ell^2_h(M^{-1})} \lesssim \mathcal{S}^h_\alpha(f, f)\,,$$

where the projection Π_h is defined by the formula $(\Pi_h f)_j = M_j\,(\sum_{k\in\mathbb{Z}} f_k\,h)/(\sum_{k\in\mathbb{Z}} M_k\,h)$.

$\mathcal{S}^h_\alpha(f, f)$ also provides a **gain of fractional Sobolev regularity**.

Proposition

For any suitably summable sequence $f = (f_j)_j$, one has:

$$\mathcal{S}^h_\alpha(f, f) \gtrsim \|(f - \Pi_h f)M^{-1/2}\|^2_{H^{\alpha/2}_h}\,.$$

A discrete interpolation inequality in weighted spaces

Proposition

There exist $\eta > 0$ *and* $h_0 > 0$ *such that for any* $h \in (0, h_0)$ *and for any* $\varepsilon \in (0, \eta)$*, there is* $K(\varepsilon) > 0$ *such that*

$$\|f\|_{\ell^2_h(M^{-1})} \|D_h f\|_{\ell^2_h(M^{-1})} + \|D_h f\|^2_{\ell^2_h(M^{-1})}$$
$$\leqslant \; K(\varepsilon) \left(\mathcal{S}^h_\alpha(f, f) \; + \; \|\Pi_h f\|^2_{\ell^2_h(M^{-1})} \right) + \varepsilon \, \mathcal{S}^h_\alpha(D_h f, D_h f)$$

where we recall that $(\Pi_h f)_j = M_j \, (\sum_{k \in \mathbb{Z}} f_k \, h)/(\sum_{k \in \mathbb{Z}} M_k \, h)$.

Long-time behavior

1. Semi-discretized homogeneous case:

$$\partial_t f_j \;=\; (L_\alpha^h f)_j\,, \quad \forall j \in \mathbb{Z}\,,$$

with some given initial data $(f_j^0)_j$.

Theorem (Ayi, Herda, Hivert, Tristani (2021))

There exists $h_0 > 0$ such that if f is a solution of the **semi-discrete Lévy-Fokker-Planck equation** *with initial data $(f_j^0)_j \in H_h^1(M^{-1})$ then, for all $t \geqslant 0$ and $h \in (0, h_0)$ one has*

$$\|f(t) - f^\infty\|_{H_h^1(M^{-1})} \;\leqslant\; C\,\|f^0 - f^\infty\|_{H_h^1(M^{-1})}\, e^{-\lambda t}$$

where

$$f^\infty := \frac{\langle f^0 \rangle_h}{\langle M \rangle_h}\, M \quad \text{with} \quad \langle f \rangle_h := \sum_{j \in \mathbb{Z}} f_j\, h$$

for some constants $C \geqslant 1$ and $\lambda > 0$ depending only on α.

2. Semi-discretized inhomogeneous case:

$$\partial_t f_{i,j} + \left(T^{\Delta x} f\right)_{i,j} = \left(L_\alpha^{\Delta v} f\right)_{i,j}, \ \forall (i,j) \in \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z},$$

The **scalar product**

$$\langle f, g \rangle_{\ell^2_{\Delta x, \Delta v}(M^{-1})} = \sum_{i \in \mathbb{Z}/N_x\mathbb{Z}} \sum_{j \in \mathbb{Z}} \frac{f_{i,j} g_{i,j}}{M_j} \Delta x \Delta v \,.$$

The **associated norm** $\| \cdot \|_{\ell^2_{\Delta x, \Delta v}(M^{-1})}$.

A **discrete** H^1 **weighted norm**

$$\|f\|^2_{H^1_{\Delta x, \Delta v}(M^{-1})} = \|f\|^2_{\ell^2_{\Delta x, \Delta v}(M^{-1})} + \|D_{\Delta x} f\|^2_{\ell^2_{\Delta x, \Delta v}(M^{-1})} + \|D_{\Delta v} f\|^2_{\ell^2_{\Delta x, \Delta v}(M^{-1})},$$

where

$$\forall (i,j) \in \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}, \ (D_{\Delta x} f)_{i,j} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}, \ \ (D_{\Delta v} f)_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta v}\,.$$

Theorem (Ayi, Herda, Hivert, Tristani (2021))

Suppose that N_x is odd. There exists $\Delta v_0 > 0$ such that if f is solution of the **semi-discrete kinetic Lévy-Fokker-Planck equation** *with initial data $(f^0_{i,j})_{i,j} \in H^1_{\Delta x,\Delta v}(M^{-1})$ then, for all $\Delta v < \Delta v_0$ and for all $t \geqslant 0$, one has*

$$\|f(t) - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})} \leqslant C\|f^0 - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})}\, e^{-\lambda t},$$

where

$$f^\infty := \frac{\left\langle f^0 \right\rangle_{\Delta x,\Delta v}}{\left\langle M \right\rangle_{\Delta x,\Delta v}} M \quad \text{with} \quad \left\langle f \right\rangle_{\Delta x,\Delta v} := \sum_{(i,j)\in \mathbb{Z}/N_x\mathbb{Z}\times\mathbb{Z}} f_{i,j}\, \Delta x \Delta v$$

for some constant $C \geqslant 1$ and $\lambda > 0$ depending only on α.

3. Fully discrete implicit in time inhomogeneous case:

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t} + \left(T^{\Delta x} f\right)_{i,j}^{n+1} = \left(L_{\alpha}^{\Delta v} f\right)_{i,j}^{n+1}, \ \forall (i,j) \in \mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}, n \in \mathbb{N}.$$

Theorem (Ayi, Herda, Hivert, Tristani (2021))

Suppose that N_x is odd. There exists $\Delta v_0 > 0$ such that if f is a solution of the discrete kinetic Lvy-Fokker-Planck equation with initial data $(f_{i,j}^0)_{i,j} \in H^1_{\Delta x, \Delta v}(M^{-1})$, then for all $\Delta v < \Delta v_0$ and for all $n \in \mathbb{N}$, one has

$$\|f^n - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})} \leqslant C\|f^0 - f^\infty\|_{H^1_{\Delta x,\Delta v}(M^{-1})} \left(1 + 2\lambda\Delta t\right)^{-\frac{n}{2}},$$

for some constants $C \geqslant 1$ and $\lambda > 0$ depending only on α. The global equilibrium f^∞ is the same as in the previous theorem.

A technical lemma

Continuous setting: $[\partial_v, L_\alpha] = \partial_v$ easily computed. **Discrete case:** more intricate.

Lemma

$$\left| \left\langle [D_h, L_\alpha^h] f, g \right\rangle_{\ell_h^2(M^{-1})} \right| \lesssim \|f\|_{\ell_h^2(M^{-1})} \|g\|_{\ell_h^2(M^{-1})} + \|D_h f\|_{\ell_h^2(M^{-1})} \|g\|_{\ell_h^2(M^{-1})} .$$

About the proof. We clearly have $[D_h, \Lambda_\alpha^h] = 0$ and

$$\begin{aligned} \left\langle [D_h, L_\alpha^h] f, g \right\rangle_{\ell_h^2(M^{-1})} &= \left\langle [D_h, \Gamma_\alpha^h] f, g \right\rangle_{\ell_h^2(M^{-1})} \\ &= \left\langle D_h \Gamma_\alpha^h f, g \right\rangle_{\ell_h^2(M^{-1})} - \left\langle \Gamma_\alpha^h D_h f, g \right\rangle_{\ell_h^2(M^{-1})} . \end{aligned}$$

$$\left\langle [D_h, L_\alpha^h] f, g \right\rangle_{\ell_h^2(M^{-1})} = C_0 + C_{-1} + C_1 + C_{-2,2}\,,$$

with

$$C_0 = \sum_{j\in\mathbb{Z}} \frac{h}{M_j} \frac{f_j g_j}{(2h)^2} \left(\frac{(VM)_{j-1/2}}{M_{j-1}} - \frac{(VM)_{j-1/2}}{M_j} + \frac{(VM)_{j+1/2}}{M_{j+1}} - \frac{(VM)_{j+1/2}}{M_j} \right).$$

Idea: Rewrite this term using the **bounds on the discrete equilibrium** M_j and its derivatives, and some **decay estimates** on (VM) and its derivatives.

$$|(D_h M)_{j+m}| \lesssim \frac{1}{\langle hj \rangle^{2+\alpha}} \quad \text{and} \quad |(D_h^+ M)_{j+m}| \lesssim \frac{1}{\langle hj \rangle^{2+\alpha}}\,.$$

$$|(D_h^+ M^{1/2})_{j+m}| \lesssim \frac{1}{\langle hj \rangle^{(3+\alpha)/2}}, \quad |(D_h^+ M^{-1/2})_{j+m}| \lesssim \langle hj \rangle^{(-1+\alpha)/2}$$

$$|(D_h^+ M^{-1})_{j+m}| \lesssim \langle hj \rangle^{\alpha}\,.$$

$$|(D_h^2 M)_{j+m}| \lesssim \frac{1}{\langle hj \rangle^{3+\alpha}} \qquad \text{and} \qquad \left| \frac{M_{j+m+1} + M_{j+m-1} - 2M_{j+m}}{h^2} \right| \lesssim \frac{1}{\langle hj \rangle^{3+\alpha}}\,.$$

$$\left|(VM)_{j+\frac{m}{2}}\right| \lesssim \frac{1}{\langle hj\rangle^{\alpha}}\,.$$

$$\frac{1}{h}\left|(VM)_{j+\frac{m}{2}} - (VM)_{j+\frac{m}{2}-1}\right| \lesssim \frac{1}{\langle hj\rangle^{1+\alpha}}\,.$$

$$\frac{1}{h^2}\left|(VM)_{j+\frac{m}{2}} - (VM)_{j+\frac{m}{2}-1} - (VM)_{j+\frac{m}{2}-2} + (VM)_{j+\frac{m}{2}-3}\right| \lesssim \frac{1}{\langle hj\rangle^{2+\alpha}}\,.$$

$$C_0 = \sum_{j\in\mathbb{Z}} \frac{h}{4M_j}\, f_j\, g_j\, \frac{(VM)_{j+1/2} - (VM)_{j-1/2}}{h}\, \frac{1}{h}\left(\frac{1}{M_{j+1}} - \frac{1}{M_j}\right)$$
$$+ \sum_{j\in\mathbb{Z}} \frac{h}{4M_j}\, f_j\, g_j\, (VM)_{j-1/2}\, \frac{1}{h^2}\left(\frac{1}{M_{j+1}} - \frac{2}{M_j} + \frac{1}{M_{j-1}}\right)\,.$$

Thanks to our bounds, the following inequality holds

$$|C_0| \lesssim \sum_{j\in\mathbb{Z}} \frac{h}{M_j}\, |f_j|\, |g_j|\,,$$

which yields $|C_0| \lesssim \|f\|_{\ell_h^2(M^{-1})}\|g\|_{\ell_h^2(M^{-1})}$ with **Cauchy-Schwarz inequality**.

Proposition

Let f be a solution to $\partial_t f_j = (L_\alpha^h f)_j\,, \quad \forall j \in \mathbb{Z}$. *Then we have:*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\ell^2_h(M^{-1})} = -\mathcal{S}^h_\alpha(f,f)$$

and there exists $C > 0$ *(depending only on* α*) such that*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D_h f\|^2_{\ell^2_h(M^{-1})} \leqslant -\mathcal{S}^h_\alpha(D_h f, D_h f) + C\|D_h f\|^2_{\ell^2_h(M^{-1})} + C\|f\|_{\ell^2_h(M^{-1})}\|D_h f\|_{\ell^2_h(M^{-1})}\,.$$

Proof.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{\ell^2_h(M^{-1})} = \left\langle L^h_\alpha f, f\right\rangle_{\ell^2_h(M^{-1})} = -\mathcal{S}^h_\alpha(f,f).$$

Moreover,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D_h f\|^2_{\ell^2_h(M^{-1})}\\ &= \left\langle D_h L^h_\alpha f, f\right\rangle_{\ell^2_h(M^{-1})} = \left\langle [D_h, L^h_\alpha] f, D_h f\right\rangle_{\ell^2_h(M^{-1})} + \left\langle L^h_\alpha D_h f, D_h f\right\rangle_{\ell^2_h(M^{-1})}\,.\end{aligned}$$

and

$$\left\langle L^h_\alpha D_h f, D_h f\right\rangle = -\mathcal{S}^h_\alpha(D_h f, D_h f)\,.$$

Proof in the homogeneous case. We introduce the functional ($a > 0$ to be chosen later):

$$\mathcal{F}(f) := \|f\|^2_{\ell^2_h(M^{-1})} + a\|D_h f\|^2_{\ell^2_h(M^{-1})}$$

$$\mathcal{F}(f) \sim \|f\|^2_{H^1_h(M^{-1})}.$$

We consider **an initial data** $(f_j^0)_j$ with **vanishing mass** and $f(t)$ is such that for $t \geqslant 0$, $\Pi_h f(t) = 0$ (**mass preservation**). Then, there exist constants $C > 0$ and $\eta > 0$ such that for any $\varepsilon \in (0, \eta)$, there is $K(\varepsilon) > 0$ such that:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f(t)) &\leqslant -\mathcal{S}^h_\alpha(f,f) - a\,\mathcal{S}^h_\alpha(D_h f, D_h f)\\ &\quad + a\,C\,\|D_h f\|^2_{\ell^2_h(M^{-1})} + a\,C\,\|f\|_{\ell^2_h(M^{-1})}\|D_h f\|_{\ell^2_h(M^{-1})}\\ &\leqslant -\mathcal{S}^h_\alpha(f,f) - a\,\mathcal{S}^h_\alpha(D_h f, D_h f)\\ &\quad + a\,C\,K(\varepsilon)\mathcal{S}^h_\alpha(f,f) + a\,C\,\varepsilon\,\mathcal{S}^h_\alpha(D_h f, D_h f)\,.\end{aligned}$$

Choosing first ε small enough so that $1 - C\,\varepsilon \geqslant 1/2$ and then a small enough so that $1 - a\,C\,K(\varepsilon) \geqslant 1/2$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f(t)) \leqslant -\mathcal{S}^h_\alpha(f,f) - a\,\mathcal{S}^h_\alpha(D_h f, D_h f)\,.$$

The discrete nonlocal Poincaré inequality implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(f(t)) \lesssim -\mathcal{F}(f(t))$$

and we can thus conclude thanks to a Gronwall type argument. □

Numerical simulations

Test case 1: convergence of the scheme in the homogeneous case.

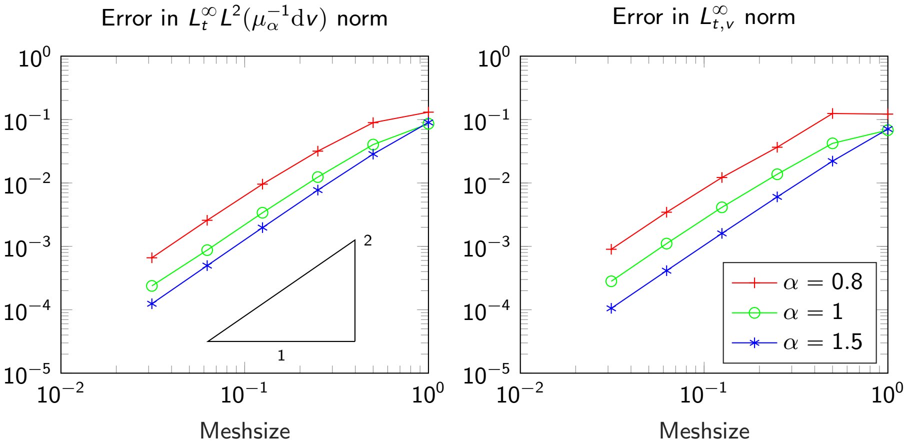

Figure: **Test case 1.** Error in $L^\infty_t L^2(\mu_\alpha^{-1}\mathrm{d}v)$ (left) and $L^\infty_{t,v}$ (right) norm between approximate and analytical solution.

Test case 2: preservation of the heavy tails (homogeneous case).
Truncated velocity domain $[-L, L]$ with $L = 20$, 1025 mesh points, time step $\Delta t = 10^{-2}$.
The initial data:

$$f(0, v) \;=\; \frac{1}{2}\chi_{[-3,-1]}(v) + \frac{1}{4}\chi_{[0,4]}(v)\,,$$

where χ_I is the indicator function of the set I.

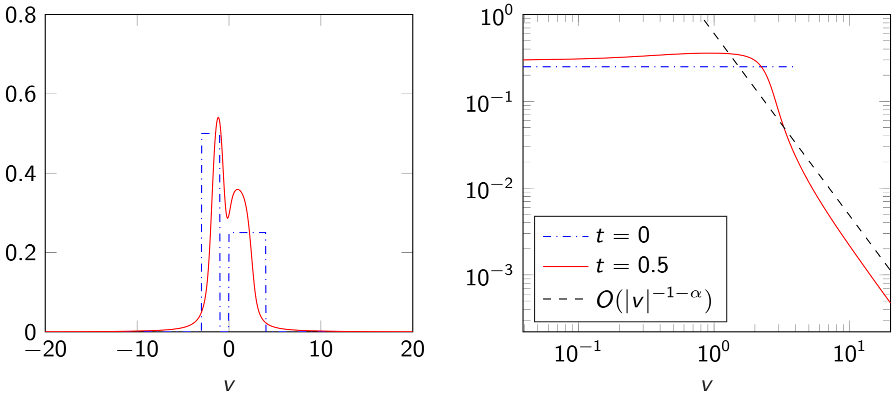

Figure: **Test case 2.** Approximate densities at $t = 0.5$. On the right the logarithmic scale allows to see the heavy-tail decay. Here $\alpha = 1.1$.

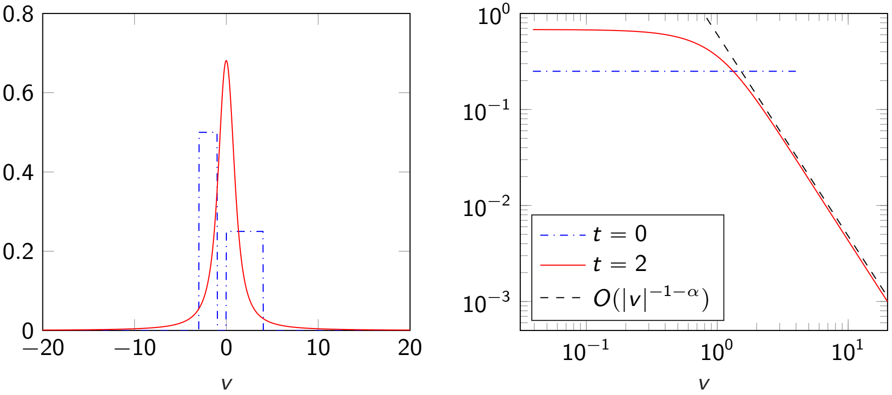

Figure: **Test case 2.** Approximate densities at $t = 2$. On the right the logarithmic scale allows to see the heavy-tail decay. Here $\alpha = 1.1$.

Test case 3: numerical hypocoercivity (long time behavior).

$$\partial_t f + v \cdot \nabla_x f = L_1 f$$

with $x \in \mathbb{R}/(2\pi\mathbb{Z})$ and $v \in \mathbb{R}$. Velocity domain truncated at $L = 16$, discretized 65 points ($J = 32$). Space domain, of size 2π, discretized with 128 points. Time step $\Delta t = 10^{-2}$, final time $T = 35$. Error of $4.5 \cdot 10^{-2}$ in $L^\infty_{t,x,v}$ norm between the computed solution and the reference solution.

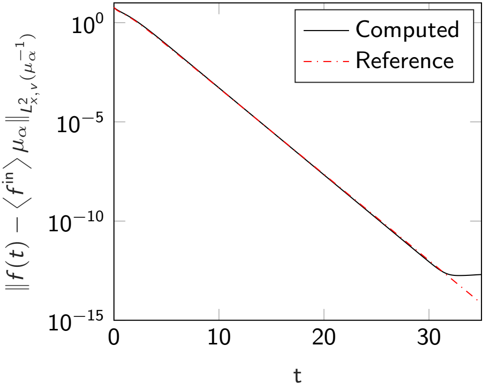

Figure: **Test case 3.** Time evolution of the distance between the steady state and the approximate and reference densities in $L^2_{x,v}(\mu_\alpha^{-1} \mathrm{d}v\mathrm{d}x)$ norm.

Thank you for your attention.