Uniform in time propagation of chaos for the generalized Dyson Brownian motion and 1D Riesz gases

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Abstract

We consider the one dimensional N-particle system in mean field interaction

$$dX_t^i = \sqrt{2\sigma_N} dB_t^i - U'\left(X_t^i\right) dt - \frac{1}{N}\sum_{j\neq i} V'(X_t^i - X_t^j)dt. \tag{1}$$

where for all $i \in \{1,...,N\}$, X_t^i denotes the position in $\mathbb{R}$ of the i-th particle, $(B_t^i)_i$ are independent Brownian motions, and σ_N is a diffusion coefficient that may depend on N. We consider singular repulsive interactions

$$\exists\alpha \geq 0,\ \forall x \in \mathbb{R},\ V'(x) = -\frac{x}{|x|^{\alpha+1}},$$

and a confining potential U such that U' is Lipschitz continuous.

The main motivating example is the (generalized) Dyson Brownian motion

$$dX_t^i = \sqrt{\frac{2\sigma}{N}} dB_t^i - \lambda X_t^i dt + \frac{1}{N}\sum_{j\neq i} \frac{1}{X_t^i - X_t^j}dt, \tag{2}$$

which holds importance in Random Matrix Theory. Equation 2 is satisfied, for $\lambda = 0$, by the eigenvalues of an $N \times N$ Hermitian matrix valued Brownian motion. For $\lambda > 0$, it corresponds to the eigenvalues of an $N \times N$ Hermitian matrix valued Ornstein-Uhlenbeck process.

The main result of this talk concerns the limit, as N goes to infinity, of (1). What we wish to prove is that *in a system of N particles in mean-field interaction, as N goes to infinity, two particles become more and more statistically independent*. This phenomenon is known as *propagation of chaos*, and we prove such a result for $\alpha \in [1, 2[$ and $\sigma_N \to 0$.

We describe a method that relies only on the well posedness of the system of particles (1) and which provides a quantitative (and in some cases uniform in time) result of propagation of chaos. We make full use of the fact that in dimension one the particle will stay ordered, and that as a consequence the interaction we consider will be convex. Using a coupling method, we prove that by taking any independent sequence of empirical measures, it is a Cauchy sequence. Then, independence ensures the fact that the limit measure is an almost surely constant random variable. This method only relies on the well posedness of the system of particles, and in particular requires no study of the non linear limit.

This is joint work with Arnaud Guillin[1] and Pierre Monmarché[2]

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